6.6 - Inverse Trigonometric Functions

Preparation (Watch at Home)

6.6a

A brief review of inverse trigonometric functions

$$\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1} x + C$$

$$\int \frac{1}{x\sqrt{x^2-4}}\,dx$$

$$= \int \frac{1}{x\sqrt{4\left(\frac{x^2}{4}-1\right)}}\,dx = \frac{1}{2}\int \frac{1}{\frac{2}{2}x\sqrt{\left(\frac{x}{2}\right)^2-1}}\,dx$$

$u = \frac{x}{2}$

$$= \frac{1}{\cancel{4}\ 2\cdot 2}\int \frac{1}{\frac{x}{2}\sqrt{\left(\frac{x}{2}\right)^2-1}}\,dx$$

$u = \frac{x}{2}$

$du = \frac{1}{2}\,dx$

$$\frac{1}{2}\int \frac{1}{u\sqrt{u^2-1}}\,du = \frac{1}{2}\sec^{-1} u + C$$

$$= \frac{1}{2}\sec^{-1}\frac{x}{2} + C$$

$$\int \frac{1}{x\sqrt{25x^2-4}}\,dx = \frac{1}{2}\int \frac{1}{x\sqrt{\left(\frac{5x}{2}\right)^2-1}}\,dx$$

$u = \frac{5}{2}x$

$u = x^2$

$du = 2x\,dx$

$\frac{du}{2x} = dx$

$du = \frac{5}{2}\,dx$

$dx = \frac{2}{5}\,du$

$y = \sin x \rightarrow x = \sin y \Rightarrow y = \sin^{-1} x$

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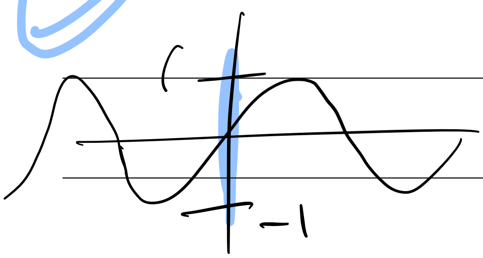
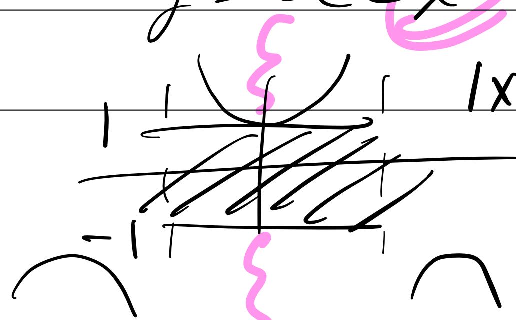


$y = \sec x$ $\qquad$ $y = \sec^{-1} x$



$|x| \geq 1$

6.6b

Calculus of inverse trigonometric functions

$\sqrt{x^2-1}$

$\sqrt{1-x^2}$

$$\frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}$$

$\log_5 125 = 3$

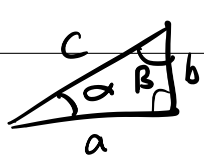
(what is the exponent

$$\frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}$$

$\sin\alpha = \frac{b}{c}$

$$\frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}$$

$\sin^{-1}\frac{b}{c} = \alpha$



$\sin^{-1}\frac{b}{c} = \alpha$ $\qquad$ $\cos^{-1}\frac{b}{c} = \beta$

$\alpha + \beta = \pi/2$

$\sin^{-1}\frac{b}{c} + \cos^{-1}\frac{b}{c} = \pi/2 \Rightarrow \sin^{-1}x + \cos^{-1}x = \pi/2$

$\Rightarrow \sin^{-1}x = \pi/2 - \cos^{-1}x$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}} \Longrightarrow \int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1} x + C$$

$$\frac{d}{dx}(\cos^{-1} x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{x^2+1} \Longrightarrow \int \frac{1}{x^2+1}\,dx = \tan^{-1} x + C$$

$$\frac{d}{dx}\left(\cot^{-1} x\right) = -\frac{1}{x^2+1}$$

$$\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{x\sqrt{x^2-1}} \Longrightarrow \int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1} x + C$$

$$\frac{d}{dx}\left(\csc^{-1} x\right) = -\frac{1}{x\sqrt{x^2-1}}$$

Note: the formula for the derivative of the inverse secant assumes a restricted domain for the secant of $\{x|\ 0 < x < \frac{\pi}{2}\ or\ \pi < x < \frac{3\pi}{2}\}$. If instead the restricted domain is $\left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$, then the differentiation rule becomes $\frac{d}{dx}\left(\sec^{-1} x\right) = \frac{1}{|x|\sqrt{x^2-1}}$.

6.6c

Exercises:

Find the derivative of the function. Simplify where possible.

$g(x) = \arccos \sqrt{x}$

$y = \cos^{-1}\left(\sin^{-1} t\right)$

$R(t) = \arcsin\left(\frac{1}{t}\right)$

6.6e

Evaluate the integral

$$\int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \frac{6}{\sqrt{1-p^2}}\, dp$$

$$\int_0^{\frac{\sqrt{3}}{4}} \frac{dx}{1 + 16x^2}$$

In-Class Discussion

6.6d

$$y = \arctan \sqrt{\frac{1 - x}{1 + x}}$$

In-Class Practice

Find the derivative of the function. Simplify where possible.

$y = \tan^{-1}(x^2)$

$$y' = \frac{2x}{1+x^4}$$

2. Find the derivative of the function. Find the domains of the function and its derivative.

$f(x) = \arcsin(e^x)$

$$f'(x) = \frac{e^x}{\sqrt{1-e^{2x}}}$$

Aside:

$$\int e^x \sin e^x \, dx$$

$u = e^x \Rightarrow du = e^x dx$

$$= -\cos e^x + C$$

Consider

$$\int \frac{e^x}{1+e^x} dx = \ln(1+e^x) + C \qquad \int \frac{e^x}{1+e^{2x}} dx = \tan^{-1} e^x + C$$

3. Find an equation of the tangent line to the curve $y = 3\arccos\left(\frac{x}{2}\right)$ at the point $(1,\ \pi)$.

More aside: $\int \frac{1}{1+x^2} dx \qquad \int \frac{x}{1+x^2} dx$

$$y' = -\frac{3}{2\sqrt{1-\frac{x^2}{4}}}$$

$$y'|_{x=1} = -\frac{3}{2\sqrt{3/4}} = -\sqrt{3}$$

$$y - \pi = -\sqrt{3}(x-1) \Rightarrow y = -\sqrt{3}x + \pi + \sqrt{3}$$

4. Find the limit.

$$\lim_{x\to\infty} \arccos\left(\frac{1+x^2}{1+2x^2}\right)$$

Evaluate the integral.

5. $\displaystyle\int \frac{(\arctan x)^2}{x^2+1}\,dx$

$u = \arctan x \Rightarrow du = \frac{1}{x^2+1}dx$

$$= \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}(\arctan x)^3 + C$$

6. ~~$\displaystyle\int \frac{1}{x\sqrt{x^2-4}}\,dx$~~

$$\int \frac{1}{x\sqrt{16x^2-9}}\,dx = \frac{1}{3}\int \frac{1}{x\sqrt{(\frac{4}{3}x)^2-1}}\,dx$$

$\sqrt{9\left(\frac{16}{9}x^2-1\right)}$

$u = \frac{4}{3}x \rightarrow x = \frac{3}{4}u$

$\rightarrow du = \frac{4}{3}dx$

so $dx = \frac{3}{4}du$

$$\frac{1}{3}\int \frac{1}{\cancel{\frac{3}{4}}u\sqrt{u^2-1}}\cancel{\frac{3}{4}}du$$

$$= \frac{1}{3}\sec^{-1}u + C = \frac{1}{3}\sec^{-1}\frac{4}{3}x + C$$

7. $\displaystyle\int \frac{1}{25+x^2}\,dx$

$$= \frac{1}{\cancel{25}_{5 \cdot 5}} \int \frac{1}{1+\left(\frac{x}{5}\right)^2} dx \qquad u = \frac{x}{5}$$

$$du = \frac{1}{5} dx$$

$$= \frac{1}{5} \int \frac{1}{1+u^2} du$$

$$= \frac{1}{5} \tan^{-1} \frac{x}{5} + C$$