6.8 - Indeterminate Forms and l'Hospital's Rule

Not indeterminate

$\frac{1}{0} \to \lim_{x\to 0} \frac{1}{x} = \infty$

Preparation (Watch at Home)

6.8a

$\frac{0}{0}$, $\frac{\infty}{\infty}$, 0^0, ∞^0,

$0\cdot\infty$, $\infty-\infty$, 1^∞

$\frac{0}{\infty} \to 0$

$\frac{\infty}{0} \to \infty$

$0^\infty = 0$

1, 2, 3, ...

$\frac{p}{q}$ $\frac{1}{2}$ $\frac{1}{3}$ $\frac{1}{4}$ $\frac{2}{2}$ $\frac{2}{3}$ $\frac{2}{4}$

0. _ _ _ _ _ _ _ _

l'Hôspital's Rule for indeterminate forms

Suppose that f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} = g(x) = 0$
or that $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} = g(x) = \pm\infty$

Then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ if the limit on the right-hand side exists or is ∞ or $-\infty$

Indeterminate forms

$\frac{0}{0}$, $\frac{\infty}{\infty}$, $0\cdot\infty$, $\infty-\infty$, 0^0, ∞^0, and 1^∞

Exercises:

Find the limit. Use l'Hôspital's Rule where appropriate. If there is a more elementary method, consider using it. If l'Hôspital's Rule doesn't apply, explain why.

6.8b

$$\lim_{x \to 0} \frac{x^2}{1 - \cos x}$$

$$\lim_{\theta \to \pi} \frac{1 + \cos \theta}{1 - \cos \theta}$$

$$\lim_{x \to \infty} \frac{\ln \sqrt{x}}{x^2}$$

$$\lim_{x \to 0} \frac{\sinh x - x}{x^3}$$

6.8d

$$\lim_{x \to -\infty} x \ln\left(1 - \frac{1}{x}\right)$$

$$\lim_{x \to 1^+} \left[\ln\left(x^7 - 1\right) - \ln\left(x^5 - 1\right)\right]$$

6.8e

$$\lim_{x \to 0^+} (\tan 2x)^x$$

l'Hospital's Rule:

If $\lim\limits_{\substack{x \to a \\ \text{or } x \to \infty}} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\infty}{\infty}$,

then $\lim\limits_{\substack{x \to a \\ x \to \infty}} \frac{f(x)}{g(x)} = \lim\limits_{\substack{x \to a \\ x \to \infty}} \frac{f'(x)}{g'(x)}$

In-Class Discussion

6.8c

$$\lim_{t \to 0} \frac{8^t - 5^t}{t}$$

Check: $\frac{1-1}{0} \to \frac{0}{0}$ ✓

$$\overset{H}{=} \lim_{t \to 0} \frac{\ln 8 \cdot 8^t - \ln 5 \cdot 5^t}{1} = \ln 8 - \ln 5 = \ln \frac{8}{5}$$

$\to \frac{0}{0}$ ✓

Note: $\lim_{x \to 0} \frac{x^2+x}{1-e^x} = \lim_{x \to} \frac{2x+1}{-e^x} = -1$

$$\lim_{x \to a^+} \frac{\cos x \ln(x-a)}{\ln\left(e^x - e^a\right)}$$

6.8f

$$\lim_{x \to \infty} \left(1 + \frac{a}{x}\right)^{bx}$$

6.8g

Proof:

l'Hôspital's Rule

Suppose that f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} = g(x) = 0$
or that $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} = g(x) = \pm\infty$

Then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ if the limit on the right-hand side exists or is ∞ or $-\infty$

Indeterminate forms

$\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty, \infty - \infty, 0^0, \infty^0$, and 1^∞

In-Class Practice

Find the limit. Use l'Hospital's Rule where appropriate. If there is a more elementary method, consider using it. If l'Hospital's Rule doesn't apply, explain why.

1. $\lim_{x \to 0} \dfrac{\tan 3x}{\sin 2x}$

2. $\lim_{u \to \infty} \frac{e^{\frac{u}{10}}}{u^3}$

3. $\lim_{x \to 0} \frac{x - \sin x}{x - \tan x}$

4. $\lim_{x \to \infty} x^{\frac{3}{2}} \sin\left(\frac{1}{x}\right)$

5. $\lim_{x \to 0^+} (\tan 2x)^x$