$u = x^2$

$\frac{du}{dx} = 2x$

$$\int x \cos(x^2)\,dx$$

du f(u) ✓

$u = x^2 \Rightarrow du = 2x\,dx \ldots$

$$\int x e^{-x^2}\,dx$$

$u = -x^2 \Rightarrow du = -2x\,dx$

$$\int x \cos x\,dx$$

g(x) f(x)

$$\int x e^{-x}\,dx$$

Recall: $\underline{\frac{d}{dx}(uv)} = u'v + uv' = \underline{v\frac{du}{dx} + u\frac{dv}{dx}}$

$$\Rightarrow \int \cancel{\frac{d}{dx}}(uv)\cancel{dx} = \int v \frac{du}{\cancel{dx}}\cancel{dx} + \int u \frac{dv}{\cancel{dx}}\cancel{dx}$$

$$uv = \int v\,du + \underline{\int u\,dv}$$

solve for $\int u\,dv$

Integration by Parts Formula

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Ex: $\int \underset{u}{x}\,\underset{dv}{\cos x\,dx} = x \sin x - \int \sin x\,dx$

$u = x \qquad dv = \cos x\,dx$

$du = dx \qquad v = \sin x$

$= x \sin x + \cos x + C$

7.1 - Integration by Parts

7.1a

ruledgap25

The integration by parts formula

$\int u\ dv = uv - \int v\ du$

Exercises:

Evaluate the integral.

7.1b

$\int ye^{0.2y}\ dy$

$u = y \quad dv = e^{1/5 y}\, dy$

$du = dy \quad v = 5e^{1/5 y}$

$5ye^{1/5 y} - \int 5e^{1/5 y}\, dy$

$5ye^{0.2y} - 25e^{0.2y} + C$

$\int \ln x\ dx$

$\frac{d}{dx}(\ln x) = \frac{1}{x}$

$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

$\frac{d}{dx}[\quad] = \ln x$

~~want~~ $dv = \cancel{\ln x\, dx}$

$u = \ln x \quad dv = dx$

$du = \frac{1}{x}dx \quad v = x$

$uv = \int v\,du = x\ln x - \int dx$

$\int x^2 \ln x\, dx$

$u = \ln x \quad dv = x^2 dx$

$du = \frac{1}{x}dx \quad v = \frac{1}{3}x^3$

$= x\ln x - x + C$

Check: $\frac{d}{dx}(x\ln x - x)$

$= \ln x + 1 - 1 = \ln x$ ✓

7.1c

$I = \int e^{-\theta}\cos 2\theta \, d\theta$

$u = e^{-\theta}$ $\quad dv = \cos 2\theta \, d\theta$

$du = -e^{-\theta}d\theta$ $\quad v = \frac{1}{2}\sin 2\theta$

$\frac{1}{2}e^{-\theta}\sin 2\theta + \int \frac{1}{2}e^{-\theta}\sin 2\theta \, d\theta$

$u = \frac{1}{2}e^{-\theta}$ $\quad dv = \sin 2\theta \, d\theta$

$du = -\frac{1}{2}e^{-\theta}d\theta$ $\quad v = -\frac{1}{2}\cos 2\theta$

$I = \frac{1}{2}e^{-\theta}\sin 2\theta - \frac{1}{4}e^{-\theta}\cos 2\theta - \frac{1}{4}\cancel{\int e^{-\theta}\cos 2\theta\, d\theta}$

$-\frac{1}{4}I$

add

$C = \frac{4}{5}C_1$

$\frac{5}{4}I = \frac{1}{2}e^{-\theta}\sin 2\theta - \frac{1}{4}e^{-\theta}\cos 2\theta + C_1$

$I = \frac{2}{5}e^{-\theta}\sin 2\theta - \frac{1}{5}e^{-\theta}\cos 2\theta + C$

7.1d

$\int_0^1 (x^2+1)\, e^{-x}\, dx$

$u = x^2+1$ $\quad dv = e^{-x}dx$

$du = 2x\,dx$ $\quad v = -e^{-x}$

$$-(x^2+1)e^{-x}\Big|_0^1 + \int_0^1 2x\, e^{-x}\,dx \qquad \begin{array}{ll} u=2x & dv=e^{-x}dx \\ du=2dx & v=-e^{-x} \end{array}$$

$$-2e^{-1}+1 \;-2xe^{-x}\Big|_0^1 + \int_0^1 2e^{-x}\,dx$$

$$-2e^{-1}+1-2e^{-1}-2e^{-x}\Big|_0^1$$

$$-4e^{-1}+1-2e^{-1}+2$$

$$3-6e^{-1} \quad \text{OR} \quad 3-\frac{6}{e}$$

7.1e

$\int_1^{\sqrt{3}} \arctan\left(\frac{1}{x}\right)\ dx$

$$u=\tan^{-1}\frac{1}{x} \qquad dv=dx$$

$$du=\frac{-\frac{1}{x^2}}{1+\frac{1}{x^2}}\,\frac{x^2}{x^2}\,dx$$

$$\int \ln x \tan^{-1} x$$

$$u=\tan^{-1}x \qquad dv=\ln x\,dx$$

$$du=\frac{1}{1+x^2}dx \qquad s=\ln x \quad dt=dx$$

$$uv-\int v\,du$$

Alternative order of events:

$$\int_0^1 (x^2+1)\, e^{-x}\, dx \qquad u = x^2+1 \quad dv = e^{-x}dx$$
$$du = 2x\,dx \quad v = -e^{-x}$$

$$= -(x^2+1)e^{-x}\Big|_0^1 + \int_0^1 2x\, e^{-x} dx \qquad \underline{u = 2x} \quad dv = e^{-x}dx$$
$$du = 2dx \quad \underline{v = -e^{-x}}$$

$$\downarrow$$

$$= \Big[-(x^2+1)e^{-x} - 2xe^{-x}\Big|_0^1 + \int_0^1 2e^{-x} dx$$

$$= \Big[-(x^2+1)e^{-x} - 2xe^{-x} - 2e^{-x}\Big|_0^1$$

Integration by Parts Formula:

$$\int u\,dv = uv - \int v\,du$$

In-Class Practice

Evaluate the integral.

$$\int u\,dv = uv - \int v\,du$$

1. $\int t^2 \sin \beta t \, dt$

$u = t^2 \qquad dv = \sin\beta t\,dt$

$du = 2t\,dt \qquad v = -\frac{1}{\beta}\cos\beta t$

$$-\frac{1}{\beta}t^2\cos\beta t + \int \frac{2}{\beta} t \cos\beta t\,dt$$

$u = \frac{2}{\beta}t \qquad dv = \cos\beta t\,dt$

$du = \frac{2}{\beta}dt \qquad v = \frac{1}{\beta}\sin\beta t$

$$-\frac{1}{\beta}t^2\cos\beta t + \frac{2}{\beta^2}t\sin\beta t - \int \frac{2}{\beta^2}\sin\beta t\,dt$$

$$-\frac{1}{\beta}t^2\cos\beta t + \frac{2}{\beta^2}t\sin\beta t + \frac{2}{\beta^3}\cos\beta t + C$$

2. $\int_1^2 w^2 \ln w \, dw$

3. $\int_1^2 \frac{(\ln x)^2}{x^3} \, dx$

$t = \ln x$ $\quad$ $t^2 \, dt$

$dt = \frac{1}{x} dx$ $\quad$ $\frac{1}{3} t^3$

$u = \frac{1}{x^2} \qquad dv = \frac{(\ln x)^2}{x} dx$

$du = -\frac{2}{x^3} dx \qquad v = \frac{1}{3} (\ln x)^3$

$\frac{1}{3x^2} (\ln x)^3 \Big|_1^2 + \int_1^2 \frac{2}{3x^3} (\ln x)^3 \, dx$

This got worse

$u = (\ln x)^2 \qquad dv = \frac{1}{x^3} dx$

$du = \frac{2 (\ln x)}{x} \qquad v = -\frac{1}{2x^2}$

$-\frac{1}{2x^2} (\ln x)^2 \Big|_1^2 + \int_1^2 \frac{\ln x}{x^3} \, dx$

$u = \ln x \qquad dv = \frac{1}{x^3} dx$

$du = \frac{1}{x} dx \qquad V = -\frac{1}{2x^2}$

$\Big(-\frac{(\ln x)^2}{2x^2} - \frac{\ln x}{2x^2} \Big|_1^2 + \int_1^2 \frac{1}{2x^3} dx$

$$\left(-\frac{\ln^2 x + \ln x}{2x^2} - \frac{1}{4x^2}\right)\Big|_1^2$$

$$-\frac{\ln^2 2 + \ln 2}{8} - \frac{1}{16} + \frac{1}{4}$$

$$\frac{3}{16} - \frac{\ln^2 2 + \ln 2}{8}$$

5. $\int e^{3\theta} \sin\theta \, d\theta = I$

$u = e^{3\theta}$ $\quad dv = \sin\theta \, d\theta$

$du = 3e^{3\theta} d\theta$ $\quad v = -\cos\theta$

$$I = -e^{3\theta}\cos\theta + \int 3e^{3\theta}\cos\theta \, d\theta$$

$u = 3e^{3\theta}$ $\quad dv = \cos\theta \, d\theta$

$du = 9e^{3\theta} d\theta$ $\quad v = \sin\theta$

$$I = -e^{3\theta}\cos\theta + 3e^{3\theta}\sin\theta - \int 9e^{3\theta}\sin\theta \, d\theta$$

$$I = -e^{3\theta}\cos\theta + 3e^{3\theta}\sin 3\theta - 9I$$

$$10I = -e^{3\theta}\cos\theta + 3e^{3\theta}\sin\theta + C_1$$

$$I = -\frac{1}{10}e^{3\theta}\cos\theta + \frac{3}{10}e^{3\theta}\sin\theta + C$$ $\leftarrow C = \frac{C_1}{10}$

$$\int 9x^3 e^{-x}\,dx = 9\int x^3 e^{-x}\,dx$$

$u = x^3 \qquad dv = e^{-x}dx$

$du = 3x^2dx \qquad v = -e^{-x}$

$$-9x^3e^{-x} + \underbrace{9\int 3x^2e^{-x}dx}_{+27\int x^2e^{-x}dx}$$

$u = x^2 \qquad dv = e^{-x}dx$

$du = 2x\,dx \qquad v = -e^{-x}$

$$-9x^3e^{-x} - 27x^2e^{-x} + \underbrace{27\int 2xe^{-x}dx}_{54\int xe^{-x}dx}$$

$u = x \qquad dv = e^{-x}dx$

$du = dx \qquad v = -e^{-x}$

$$-9x^3e^{-x} - 27x^2e^{-x} - 54xe^{-x} + 54\int e^{-x}dx$$

$$-9e^{-x}(x^3+3x^2+6x+6) + C$$

OR : $\int 9x^3 e^{-x}\, dx$

$u = 9x^3 \qquad dv = e^{-x}\,dx$

$du = 27x^2\,dx \qquad v = -e^{-x}$

$-9x^3 e^{-x} + \int 27x^2 e^{-x}\, dx$

$u = 27x^2 \qquad dv = e^{-x}\,dx$

$du = 54x\,dx \qquad v = -e^{-x}$

$-9x^3 e^{-x} - 27x^2 e^{-x} + \int 54x e^{-x}\, dx$