7.2 - Trigonometric Integrals

7.2a

Exercise:

Evaluate the integral.

odd

$$\int \sin^3\theta \cos^4\theta \, d\theta$$

Rewrite $\sin^3\theta = \sin^2\theta \sin\theta$

Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$

$\Rightarrow \sin^2\theta = 1 - \cos^2\theta$

$$-\int (1-\cos^2\theta)\cos^4\theta(-\sin\theta)\,d\theta$$

$u = \cos\theta \qquad du = -\sin\theta\, d\theta$

$$-\int(\cos^4\theta - \cos^6\theta)(-\sin\theta)\,d\theta = -\int(u^4 - u^6)\,du$$

$$= -\left(\frac{1}{5}u^5 - \frac{1}{7}u^7\right) + C = \frac{1}{7}\cos^7\theta - \frac{1}{5}\cos^5\theta + C$$

Pythagorean Identities

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$1 + \cot^2\theta = \csc^2\theta$

7.2b

Exercise:

Evaluate the integral.

Recall:

$\sin 2\theta = 2\sin\theta\cos\theta$ (double-angle formula)

$\frac{1}{2}\sin 2\theta = \sin\theta\cos\theta$

Power-reducing formulas

$$\int_0^{\pi} \sin^2 t \cos^4 t \, dt$$

$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$

$$\int_0^{\pi} \boxed{\sin^2 t \cos^2 t} \cos^2 t \, dt$$

$$= \int_0^{\pi} \frac{1}{4} \sin^2 2t \cos^2 t \, dt$$

$$= \int_0^{\pi} \frac{1}{4} \sin^2 2t \cdot \frac{1}{2}(1+\cos 2t) \, dt$$

$u = \sin 2t \Rightarrow du = 2\cos 2t \, dt$

$$= \frac{1}{8} \int_0^{\pi} (\sin^2 2t + \sin^2 2t \cos 2t) \, dt$$

$$= \frac{1}{8} \int_0^{\pi} \left[\frac{1}{2}(1-\cos 4t) + \underbrace{\frac{1}{2}}_{\frac{1}{2}} \underbrace{(\sin^2 2t)}_{u^2} \underbrace{(2\cos 2t)}_{du} \right] dt$$

$$= \left[\frac{1}{16}\left(t - \frac{1}{4}\sin 4t\right) + \frac{1}{48}\sin^3 2t \right]_0^{\pi} = \frac{\pi}{16}$$

Power-reducing formulas

$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$

$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

The double-angle formula
$\sin 2\theta = 2\sin\theta\cos\theta$

Exercises:

Evaluate the integral.

7.2c

$$\int \frac{\sin^2\left(\frac{1}{t}\right)}{t^2}\,dt$$

we don't have this du

$u = \tan x \Rightarrow du = \sec^2 x\,dx$

$$\int \tan^2 x \cos^3 x\,dx$$

$\tan^2 x = \sec^2 x - 1$

$\int (\sec^2 x - 1)\cos^3 x\,dx$ will work a la #1 above

$\int \frac{\sin^2 x}{\cancel{\cos^2 x}} \cos^{\cancel{3}} x\,dx = \int \sin^2 x \cos x\,dx$

$u = \sin x \quad du = \cos x\,dx$

$= \frac{1}{3}\sin^3 x + C \qquad \int u^2\,du$

7.2d

$$\int x \sin^3 x\,dx$$

7.2e

even on secant

$$\int_0^{\frac{\pi}{4}} \sec^6 x \tan^6 x \, dx$$

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

Recall: $\sec^2 x = \tan^2 x + 1$

$$\Rightarrow \sec^4 x = \tan^4 x + 2\tan^2 x + 1$$

$$\int_0^{\pi/4} \sec^4 x \tan^6 x \sec^2 x \, dx$$

$$= \int_0^{\pi/4} (\tan^4 x + 2\tan^2 x + 1) \tan^6 x \sec^2 x \, dx$$

$u = \tan x \leftarrow du = \sec^2 x \, dx$

$$= \int_0^{\pi/4} (\tan^{10} x + 2\tan^8 x + \tan^6 x) \sec^2 x \, dx$$

$$= \left(\frac{1}{11}\tan^{11} x + \frac{2}{9}\tan^9 x + \frac{1}{7}\tan^7 x\right)\Big|_0^{\pi/4} = \frac{316}{693}$$

$$\int_0^1 (u^{10} + 2u^8 + u^6) \, du$$

x	$u = \tan x$
$\pi/4$	1
0	0

odd

$$\int \tan^5 x \sec^3 x \, dx$$

odd power on tangent, $u = \sec x$

$du = \sec x \tan x \, dx$

$$\int \tan^4 x \sec^2 x \sec x \tan x \, dx$$

$$\tan^2 x + 1 = \sec^2 x \Rightarrow \tan^2 x = \sec^2 x - 1$$

$$\int (\sec^4 x - 2\sec^2 x + 1) \sec^2 x \sec x \tan x \, dx$$

$$= \int (\sec^6 x - 2\sec^4 x + \sec^2 x) \sec x \tan x \, dx$$

7.2f

$$= \int (u^6 - 2u^4 + u^2) \, du = \frac{1}{7}u^7 - \frac{2}{5}u^5 + \frac{1}{3}u^3 + C$$

Useful strategies

$$= \frac{1}{7}\sec^7 x - \frac{2}{5}\sec^5 x + \frac{1}{3}\sec^3 x + C$$

For $\int \sin^m \theta \cos^n \theta \, d\theta$:

Odd power
that's du

du is here

$$\sin^3\theta = \sin^2\theta \sin\theta = (1 - \cos^2\theta)\sin\theta$$

If $u = \cos\theta$, $du = -\sin\theta \, d\theta$

If $u = \sin\theta$, $du = \cos\theta \, d\theta$

Even powers
power-reducing formula

For $\int \tan^m \theta \sec^n \theta \, d\theta$

Even power on secant
use $\sec^2\theta = du$

$\hookrightarrow$ $u = \tan\theta$ Replace: $\sec^2 = \tan^2\theta + 1$

$u = \sec\theta$

Odd power on tangent
use $\sec\theta\tan\theta = du$

Replace: $\tan^2\theta = \sec^2\theta - 1$

Two useful integration rules

$$\int \tan x \, dx = \ln|\sec x| + C$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

$\int \sin x \cos x \, dx$

$u = \sin x \quad du = \cos x \, dx$

$\int u \, du = \frac{1}{2} u^2 + C$

$= \frac{1}{2} \sin^2 x + C$

$u = \cos x \quad du = -\sin x \, dx$

$-\int u \, du$

$= -\frac{1}{2} u^2 + C$

$= -\frac{1}{2} \cos^2 x + C$

But $\cos^2 x = 1 - \sin^2 x$

$\Rightarrow -\frac{1}{2} \cos^2 x = \underbrace{-\frac{1}{2} + \frac{1}{2} \sin^2 x}$

$\frac{1}{2} \sin^2 x - \frac{1}{2} + C$

Consider $\int \tan x \, dx = -\int \frac{-\sin x}{\cos x} dx$

$u = \cos x \Rightarrow du = -\sin x \, dx$

$= -\int \frac{du}{u} = -\ln|u| + C$

$= - \ln|\cos x| + C$ exponent

- $\int \tan x \, dx = \ln|\sec x| + C$

$$\int \sec x \, \frac{\sec x + \tan x}{\sec x + \tan x} \, dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x} \, dx$$

$u = \sec x + \tan x$

$du = (\sec x \tan x + \sec^2 x) \, dx$

$$= \ln|\sec x + \tan x| + C$$