5.4 – Differential Equations

Definitions: A **differential equation** is an equation that contains the derivatives of one or more unknown functions.

The **order** of a differential equation is the order of the highest derivative it contains.

A differential equation of the form $y' = ay$ has a **general solution** of the form $y = ce^{ax}$.

A condition which specifies the value of the general solution at a point is called an **initial condition**, and the problem of solving a differential equation subject to an initial condition is called an **initial-value problem**.

A **constant coefficient first-order homogeneous linear system** has the form

$$\begin{aligned}
y_1' &= a_{11}y_1 + a_{12}y_2 + \ldots + a_{1n}y_n \\
y_2' &= a_{21}y_1 + a_{22}y_2 + \ldots + a_{2n}y_n \\
\vdots & \qquad \vdots \qquad \vdots \qquad \vdots \\
y_n' &= a_{n1}y_1 + a_{n2}y_2 + \ldots + a_{nn}y_n
\end{aligned}$$

where $y_i = f_i(x)$ are functions to be determined, and the a_{ij}'s are constants.

This can be written in matrix notation as

$$\begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

or $\mathbf{y}' = A\mathbf{y}$.

The **trivial solution** to the above differential equation is $y_1 = y_2 = \ldots = y_n = 0$.

Example:

a. Solve the system of differential equations.

$y_1' = 2y_1 + 3y_2$

$y_2' = 2y_1 + y_2$

b. Find the solution that satisfies the initial conditions $y_1(0) = 25$, $y_2(0) = 5$.

#7 Sometimes it is possible to solve a single higher-order linear differential equation with constant coefficients by expressing it as a system and applying the methods of this section. For the differential equation $y'' - y' - 6y = 0$, show that the substitutions $y_1 = y$ and $y_2 = y'$ lead to the system

$$y_1' = y_2$$
$$y_2' = 6y_1 + y_2$$

Solve this system, and use the result to solve the original differential equation.