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1.1 – Introduction to Systems of Linear Equations

Definition: A **linear equation** can be expressed in the form $a_1x_1 + a_2x_2 + \ldots + a_nx_n = b$ where a_i are constants, not all zero, and b is a constant.

#14 In each part, use parametric equations to describe the solution set of the linear equation.

a. $x + 10y = 2$

b. $x_1 + 3x_2 - 12x_3 = 3$

c. $4x_1 + 2x_2 + 3x_3 + x_4 = 20$

d. $v + w + x - 5y + 7z = 0$

a. $x = -10y + 2$

Let $t = y$

Then $\boxed{\begin{aligned} x &= -10t + 2 \\ y &= t \end{aligned}}$

c. $x_1 = -\frac{1}{2}x_2 - \frac{3}{4}x_3 - \frac{1}{4}x_4 + 5$

Let $r = x_2$, $s = x_3$, $t = x_4$

$x_1 = -\frac{1}{2}r - \frac{3}{4}s - \frac{1}{4}t + 5$

$x_2 = r$

$x_3 = s$

$x_4 = t$

Definition: A **parameter** (in this context) is a variable that when assigned a value provides a specific solution.

Definitions: A **homogeneous linear equation** is a linear equation in which $b = 0$.
A **system of linear equations** is a finite set of linear equations. It has the form

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m \end{array}$$

Note that the first of the double subscripts references the equation number, and the second references the term in that equation.

Definition: The **augmented matrix** for the above linear system is

$$\begin{array}{ccccc} & & & & \\ \end{array}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix} \text{ or } \left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

$x_1 \quad x_2 \quad\quad x_n \quad \text{Const}$

#16 Each linear system has infinitely many solutions. Use parametric equations to describe its solution set.

a. $\begin{aligned} 6x_1 + 2x_2 &= -8 \\ 3x_1 + x_2 &= -4 \end{aligned}$

Elimination
$E_1 - 2E_2$

$$\begin{array}{rcl} 6x_1 + 2x_2 &=& -8 \\ -6x_1 - 2x_2 &=& 8 \\ \hline 0 &=& 0 \end{array}$$

$$\left[\begin{array}{cc|c} 6 & 2 & -8 \\ 3 & 1 & -4 \end{array}\right] \xrightarrow{R_2 \to R_1 - 2R_2} \left[\begin{array}{cc|c} 6 & 2 & -8 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{array}{ccc} 6 & 2 & -8 \\ -6 & -2 & 8 \\ \hline 0 & 0 & 0 \end{array}$$

$R_1: 6x_1 + 2x_2 = -8 \Rightarrow x_1 = -\frac{1}{3}x_2 - \frac{4}{3}$

Let $t = x_2$: $\boxed{\begin{aligned} x_1 &= -\tfrac{1}{3}t - \tfrac{4}{3} \\ x_2 &= t \end{aligned}}$

$$
\begin{aligned}
&2x - y + 2z = -4\\
\text{b.}\quad &6x - 3y + 6z = -12\\
&-4x + 2y - 4z = 8
\end{aligned}
\qquad
\left[\begin{array}{ccc|c} 2 & -1 & 2 & -4 \\ 6 & -3 & 6 & -12 \\ -4 & 2 & -4 & 8 \end{array}\right]
$$

$R_2 \to R_2 - 3R_1$
$R_3 \to R_3 + 2R_1$

$$\left[\begin{array}{ccc|c} 2 & -1 & 2 & -4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

R_1: $x = \frac{1}{2}y - z - 2$

$$\boxed{\begin{aligned} x &= \tfrac{1}{2}s - t - 2 \\ y &= s \\ z &= t \end{aligned}}$$

Definitions: The following are **elementary row operations** performed on a matrix.

1. Multiply a row by a nonzero constant.
2. Interchange two rows.
3. Add a constant times one row to another.

Definitions: A **solution** of a linear system is a sequence of n numbers $s_1, s_2, \dots, s_n$ that when substituted for corresponding unknowns x_i makes each equation a true statement. If $n = 2$, then the solution is an **ordered pair**, and if $n = 3$, it is an **ordered triple**. In general, an **ordered n-tuple** has the form $(s_1, s_2, \dots, s_n)$. A linear system is **consistent** if it has at least one solution. A linear system is **inconsistent** if it has no solutions.

#**19** Find all values of k for which the given augmented matrix corresponds to a consistent linear system.

a. $\left[\begin{array}{cc|c} 1 & k & -4 \\ 4 & 8 & 2 \end{array}\right]$

$R_2 \to R_2 - 4R_1$

$$\begin{array}{ccc} 4 & 8 & 2 \\ -4 & -4k & 16 \\ \hline 0 & 8-4k & 18 \end{array}$$

$\left[\begin{array}{cc|c} 1 & k & -4 \\ 0 & 8-4k & 18 \end{array}\right]$

$R_2: (8-4k)y = 18$

If $8-4k=0$

If $k \neq 2$, the system is consistent

b. $\left[\begin{array}{cc|c} 1 & k & -1 \\ 4 & 8 & -4 \end{array}\right]$

$R_2 \to R_2 - 4R_1$

$$\begin{array}{ccc} 4 & 8 & -4 \\ -4 & -4k & 4 \\ \hline 0 & 8-4k & 0 \end{array}$$

Then $R_2 \to \frac{1}{4} R_2$

$\left[\begin{array}{cc|c} 1 & k & -1 \\ 0 & 2-k & 0 \end{array}\right]$

$R_2: (2-k)y = 0$

This is consistent for $k \in R$. $\mathbb{R}$

Example: A 3-7-9 diet calls for 3 units of fat, 7 units of protein, and 9 units of carbs in each meal. Suppose an individual has three possible foods to choose from to meet these requirements. Each ounce of the food contains

x Food 1: 3 units of fat, 4 units of protein, 1 unit of carbs
y Food 2: 2 units of fat, 5 units of protein, 3 units of carbs
z Food 3: 4 units of fat, 1 unit of protein, 2 units of carbs

Let x, y, and z denote the number of ounces of the first, second, and third foods that a person will consume at the main meal. Find a linear system in x, y, and z whose solution tells how many ounces of each food must be consumed to meet the diet requirements.

fat: $3x + 2y + 4z = 3$

protein: $4x + 5y + z = 7$

carb: $x + 3y + 2z = 9$