

1.3 – Matrices and Matrix Operations

Definitions: A **matrix** is a rectangular array of numbers. The numbers in the array are called the **entries** of the matrix. The **size** of a matrix that has m rows and n columns is $m \times n$ (read “ m by n ”).

#6 Use the following matrices to compute the indicated expression if it is defined.

$$A = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 4 & 2 \\ 3 & 1 & 5 \end{bmatrix}, D = \begin{bmatrix} 1 & 5 & 2 \\ -1 & 0 & 1 \\ 3 & 2 & 4 \end{bmatrix}, E = \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix}$$

- $(2D^T - E)A$
- $(4B)C + 2B$
- $(-AC)^T + 5D^T$
- $(BA^T - 2C)^T$
- $B^T(CC^T - A^TA)$
- $D^TE^T - (ED)^T$

Definition: If A is any $m \times n$ matrix, then the **transpose of A** , denoted by A^T , is defined to be the $n \times m$ matrix that results by interchanging the rows and columns of A ; that is, the first column of A^T is the first row of A , the second column of A^T is the second row of A , and so forth.

↙ scalar multiplication

$$D^T = \begin{bmatrix} 1 & -1 & 3 \\ 5 & 0 & 2 \\ 2 & 1 & 4 \end{bmatrix}$$

$$2D^T = \begin{bmatrix} 2 & -2 & 6 \\ 10 & 0 & 4 \\ 4 & 2 & 8 \end{bmatrix}$$

$$2D^T - E = \begin{bmatrix} 2 & -2 & 6 \\ 10 & 0 & 4 \\ 4 & 2 & 8 \end{bmatrix} - \begin{bmatrix} 6 & 1 & 3 \\ -1 & 1 & 2 \\ 4 & 1 & 3 \end{bmatrix} = \begin{bmatrix} -4 & -3 & 3 \\ 11 & -1 & 2 \\ 0 & 1 & 5 \end{bmatrix}$$

$$(2D^T - E)A = \begin{bmatrix} -4 & -3 & 3 \\ 11 & -1 & 2 \\ 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$$

$$\begin{matrix} 3 \times 3 & & 3 \times 2 \end{matrix}$$

these have to match

$$= \begin{bmatrix} -12+3+3 & 0-6+3 \\ 33+1+2 & 0-2+2 \\ 0-1+5 & 0+2+5 \end{bmatrix} = \begin{bmatrix} -6 & -3 \\ 36 & 0 \\ 4 & 7 \end{bmatrix}$$

3×2

Relabel the above matrices:

$$\text{Let } A = \begin{bmatrix} -4 & -3 & 3 \\ 11 & -1 & 2 \\ 0 & 1 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix}$$

We found the product AB using the row-column rule.

Row vectors of A are $\vec{r}_1 = [-4 \ -3 \ 3]$
 $\vec{r}_2 = [1 \ -1 \ 2]$
 $\vec{r}_3 = [0 \ 1 \ 5]$
 (or row matrices)

Column vectors of B are $\vec{c}_1 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$, $\vec{c}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$
 (or column matrices)

AB can also be found using the column method

$$AB = \left[A\vec{c}_1 \mid A\vec{c}_2 \right] \quad \text{this is a partitioned matrix}$$

$$\begin{array}{cc} A\vec{c}_1 & = \begin{bmatrix} -4 & -3 & 3 \\ 11 & -1 & 2 \\ 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 36 \\ 4 \end{bmatrix} \dots \\ 3 \times 3 & 3 \times 1 \qquad \qquad \qquad 3 \times 1 \end{array}$$

We can also use the row method.

$$AB = \begin{bmatrix} \vec{r}_1 & B \\ \vec{r}_2 & B \\ \vec{r}_3 & B \end{bmatrix}$$

$$\begin{bmatrix} -4 & -3 & 3 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ -1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} -6 & -3 \end{bmatrix} \dots$$

#17 Use the column-row expansion of AB to express this product as a sum of matrix products.

$$A = \begin{bmatrix} 4 & -3 \\ 2 & -1 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 & 2 \\ -2 & 3 & 1 \end{bmatrix}$$

Definitions: The **column-row expansion** of AB is $AB = \mathbf{c}_1\mathbf{r}_1 + \mathbf{c}_2\mathbf{r}_2 + \dots + \mathbf{c}_r\mathbf{r}_r$, where $\mathbf{c}_i$ are column vectors of A and $\mathbf{r}_i$ are row vectors of B .

$$\begin{aligned} AB &= \begin{bmatrix} 4 \\ 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} + \begin{bmatrix} -3 \\ -1 \end{bmatrix} \begin{bmatrix} -2 & 3 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 4 & 8 \\ 0 & 2 & 4 \end{bmatrix} + \begin{bmatrix} 6 & -9 & -3 \\ 2 & -3 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 6 & -5 & 5 \\ 2 & -1 & 3 \end{bmatrix} \end{aligned}$$

The linear system

$$\begin{aligned} \underline{a_{11}}x_1 + \underline{a_{12}}x_2 + \dots + \underline{a_{1n}}x_n &= b_1 \\ \underline{a_{21}}x_1 + \underline{a_{22}}x_2 + \dots + \underline{a_{2n}}x_n &= b_2 \\ \vdots & \quad \quad \quad \vdots \\ \underline{a_{m1}}x_1 + \underline{a_{m2}}x_2 + \dots + \underline{a_{mn}}x_n &= b_m \end{aligned}$$

Can be expressed using matrix multiplication.

Definition: When a linear system is written using a matrix product

$$\begin{bmatrix} \underline{a_{11}} & \underline{a_{12}} & \cdots & \underline{a_{1n}} \\ \underline{a_{21}} & \underline{a_{22}} & \cdots & \underline{a_{2n}} \\ \vdots & \vdots & & \vdots \\ \underline{a_{m1}} & \underline{a_{m2}} & \cdots & \underline{a_{mn}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

This is a matrix equation

or $Ax = b$, A is called the **coefficient matrix** of the system.

#13 a. Express the matrix equation as a system of linear equations.

$$\begin{bmatrix} 5 & 6 & -7 \\ -1 & -2 & 3 \\ 0 & 4 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 5x_1 + 6x_2 - 7x_3 \\ -x_1 - 2x_2 + 3x_3 \\ 4x_2 - x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$$
$$\begin{aligned} 5x_1 + 6x_2 - 7x_3 &= 2 \\ -x_1 - 2x_2 + 3x_3 &= 0 \\ 4x_2 - x_3 &= 3 \end{aligned}$$

Definition: Two matrices are defined to be **equal** if they have the same size and their corresponding entries are equal.

Definition: If $A_1, A_2, \dots, A_r$ are matrices of the same size, and if $c_1, c_2, \dots, c_r$ are scalars, then an expression of the form $c_1A_1 + c_2A_2 + \dots + c_rA_r$ is called a **linear combination** of $A_1, A_2, \dots, A_r$ with **coefficients** $c_1, c_2, \dots, c_r$.

In #13, we have $A = \begin{bmatrix} 5 & 6 & -7 \\ -1 & -2 & 3 \\ 0 & 4 & 1 \end{bmatrix}$,

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}.$$

Note that on the left-hand side, we have

$$\begin{bmatrix} 5x_1 + 6x_2 - 7x_3 \\ -x_1 - 2x_2 + 3x_3 \\ 4x_2 - x_3 \end{bmatrix} = \begin{bmatrix} 5x_1 \\ -x_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 6x_2 \\ -2x_2 \\ +4x_2 \end{bmatrix} + \begin{bmatrix} -7x_3 \\ 3x_3 \\ -x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 6 \\ -2 \\ 4 \end{bmatrix} x_2 + \begin{bmatrix} -7 \\ 3 \\ -1 \end{bmatrix} x_3$$

#22 Example 6 of section 1.2 presents the linear system

$$\begin{array}{ccccccccc} x_1 & +3x_2 & -2x_3 & & & +2x_5 & & = & 0 \\ 2x_1 & +6x_2 & -5x_3 & -2x_4 & +4x_5 & -3x_6 & = & 0 \\ & & 5x_3 & +10x_4 & & +15x_6 & = & 0 \\ 2x_1 & +6x_2 & & +8x_4 & +4x_5 & +18x_6 & = & 0 \end{array}$$

and the reduced row echelon form of its associated augmented matrix as

$$\begin{array}{l} \begin{bmatrix} \overset{x_1}{1} & \overset{x_2}{3} & \overset{x_3}{0} & \overset{x_4}{4} & \overset{x_5}{2} & \overset{x_6}{0} & | & 0 \\ 0 & 0 & \underline{1} & 2 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & \underline{1} & | & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & | & 0 \end{bmatrix} \end{array} \begin{array}{l} \rightarrow x_1 = -3x_2 - 4x_4 - 2x_5 \\ \rightarrow x_3 = -2x_4 \\ \rightarrow x_6 = 0 \end{array}$$

Express the solution as a linear combination of column vectors that contain only numerical entries.

Let $r = x_2$, $s = x_4$, $t = x_5$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} -3r - 4s - 2t \\ r \\ -2s \\ s \\ t \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} r + \begin{bmatrix} -4 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} s + \begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} t$$

Theorem 1.3.1 If A is an $m \times n$ matrix, and if $\mathbf{x}$ is an $n \times 1$ column vector, then the product $A\mathbf{x}$ can be expressed as a linear combination of the column vectors of A in which the coefficients are the entries of $\mathbf{x}$.

See the result just before #22 above.
And see #22.

Definition: The **main diagonal** of a square matrix contains entries from the upper left to lower right corners.

Definition: If A is a square matrix, then the **trace of A** , denoted by $tr(A)$, is defined to be the sum of the entries on the main diagonal of A . The trace of A is undefined if A is not a square matrix.

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