1.6 – More on Linear Systems and Invertible Matrices

Theorem 1.6.1: A system of linear equations has zero, one, or infinitely many solutions. There are no other possibilities.

Since we know a system can have no or one solution, we'll establish that if there is more than 1 solution, there are infinitely many solutions. If $A\vec{x}=\vec{b}$ has solutions $\vec{x}_1, \vec{x}_2$, then $A\vec{x}_1 - A\vec{x}_2 = \vec{b}-\vec{b}=\vec{0}$. Let $\vec{x}_0 = \vec{x}_2 - \vec{x}_1 \Rightarrow A\vec{x}_0 = \vec{0}$

If k is any scalar, then $A(\vec{x}_1 + k\vec{x}_0) = A\vec{x}_1 + A(k\vec{x}_0)$

$$= A\vec{x}_1 + kA\vec{x}_0 = \vec{b}+\vec{0} = \vec{b}$$

Infinitely many choices for k gives infinitely many solutions.

Theorem 1.6.2: If A is an invertible $n \times n$ matrix, then for every $n \times 1$ matrix **b**, the system of equations $A\mathbf{x} = \mathbf{b}$ has exactly one solution, namely $\mathbf{x} = A^{-1}\mathbf{b}$.

Because A^{-1} is unique.

#4 Solve the system by inverting the coefficient matrix and using Theorem 1.6.2.

$$5x_1 + 3x_2 + 2x_3 = 4$$
$$3x_1 + 3x_2 + 2x_3 = 2$$
$$x_2 + x_3 = 5$$

$$A = \begin{bmatrix} 5 & 3 & 2 \\ 3 & 3 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

Inversion algorithm

$$\left[\begin{array}{ccc|ccc} 5 & 3 & 2 & 1 & 0 & 0 \\ 3 & 3 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 0 \\ 0 & 1 & 0 & -3/2 & 5/2 & -2 \\ 0 & 0 & 1 & 3/2 & -5/2 & 3 \end{array}\right]$$

$$A\vec{x} = \vec{b} \Rightarrow \vec{x} = A^{-1}\vec{b}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 & 0 \\ -3/2 & 5/2 & -2 \\ 3/2 & -5/2 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ -11 \\ 16 \end{bmatrix}$$

$$(x_1, x_2, x_3) = (1, -11, 16)$$

OR $$\vec{x} = \begin{bmatrix} 1 \\ -11 \\ 16 \end{bmatrix}$$

#11 Solve the linear systems. Using the given values for the b's solve the systems together by reducing an appropriate augmented matrix to reduced row echelon form.

$$4x_1 - 7x_2 = b_1$$
$$x_1 + 2x_2 = b_2$$

i. $b_1 = 0$, $b_2 = 1$

ii. $b_1 = -4$, $b_2 = 6$

iii. $b_1 = -1$, $b_2 = 3$

iv. $b_1 = -5$, $b_2 = 1$

$$A = \begin{bmatrix} 4 & -7 \\ 1 & 2 \end{bmatrix}$$

$$\cancel{\left[\begin{array}{cc|c} 4 & -7 & 0 \\ 1 & 2 & 1 \end{array}\right]} \qquad \cancel{\left[\begin{array}{cc|c} 4 & -7 & -4 \\ 1 & 2 & 6 \end{array}\right]}$$

$$\left[\begin{array}{cc|c} 4 & -7 & b_1 \\ 1 & 2 & b_2 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & \frac{2}{15}b_1 + \frac{7}{15}b_2 \\ 0 & 1 & -\frac{1}{15}b_1 + \frac{4}{15}b_2 \end{array}\right]$$

i) $(x_1, x_2) = \left(\frac{7}{15}, \frac{4}{15}\right)$

ii) $(x_1, x_2) = \left(-\frac{8}{15} + \frac{42}{15}, \frac{4}{15} + \frac{24}{15}\right) = \left(\frac{34}{15}, \frac{28}{15}\right)$

and so on.

OR we could have: $\left[\begin{array}{cc|cccc} 4 & -7 & 0 & -4 & -1 & -5 \\ 1 & 2 & 1 & 6 & 3 & 1 \end{array}\right]$

and row reduced.

#17 Determine conditions on the b_i's, if any, in order to guarantee that the linear system is consistent.

$$\begin{aligned} x_1 \quad -x_2 \quad +x_3 \quad +2x_4 &= b_1 \\ -2x_1 \quad +x_2 \quad +5x_3 \quad +x_4 &= b_2 \\ -3x_1 \quad +2x_2 \quad +2x_3 \quad -x_4 &= b_3 \\ 4x_1 \quad -3x_2 \quad +x_3 \quad +3x_4 &= b_4 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & -1 & 1 & 2 & b_1 \\ -2 & 1 & 5 & 1 & b_2 \\ -3 & 2 & 2 & -1 & b_3 \\ 4 & -3 & 1 & 3 & b_4 \end{array}\right]$$

reduces to
$$\left[\begin{array}{cccc|c} 1 & 0 & -8 & -3 & -b_1-b_2 \\ 0 & 1 & -11 & -5 & -2b_1-b_2 \\ 0 & 0 & 0 & 0 & b_1-b_2+b_3 \\ 0 & 0 & 0 & 0 & -2b_1+b_2+b_4 \end{array}\right]$$

This is consistent if $b_1-b_2+b_3=0$
$-2b_1+b_2+b_4=0$

This becomes $\left[\begin{array}{cccc|c} 1 & -1 & 1 & 0 & 0 \\ -2 & 1 & 0 & 1 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & -1 & -1 & 0 \\ 0 & 1 & -2 & -1 & 0 \end{array}\right]$

The system is consistent if $b_1 = b_3 + b_4$
and $b_2 = 2b_3 + b_4$.

Theorem 1.6.4 Equivalent Statements (extends Theorem 1.5.3)
If A is an $n \times n$ matrix, then the following are equivalent.
a) A is invertible.
b) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
c) The reduced row echelon form of A is I_n.
d) A is expressible as a product of elementary matrices.
e) $A\mathbf{x} = \mathbf{b}$ is consistent for every $n \times 1$ matrix $\mathbf{b}$.
f) $A\mathbf{x} = \mathbf{b}$ has exactly one solution for every $n \times 1$ matrix $\mathbf{b}$.

Recall Thm 1.4.6: If A & B are invertible, then AB is invertible. $(AB)^{-1} = B^{-1}A^{-1}$

Proved by showing that $(AB)(B^{-1}A^{-1}) = I$.

Theorem 1.6.5 Let A and B be square matrices of the same size. If AB is invertible, then A and B must also be invertible.

pf: Let $\vec{x}_0$ be a solution of $B\vec{x} = \vec{0}$.

Then $(AB)\vec{x}_0 \underset{\substack{\uparrow \\ \text{associativity of} \\ \text{matrix mult.}}}{=} A(B\vec{x}_0) = A\vec{0} = \vec{0}$

Since AB is invertible, $(AB)\vec{x} = \vec{0}$ has only the trivial solution $\Rightarrow \vec{x}_0 = \vec{0}$.

Then $B\vec{x} = \vec{0}$ has only the trivial solution.

Thus B is invertible. (B^{-1} exists)

$$A = A(BB^{-1}) = (AB)B^{-1}$$

Since (AB) and B^{-1} are invertible, their product is invertible $\Rightarrow A$ is invertible.

$$(AB)^{-1} = B^{-1}A^{-1}$$

1.4.6 together with 1.6.5 give

$$A \,\&\, B \text{ are invertible} \iff AB \text{ is invertible}$$

$$\xrightarrow{1.4.6}$$

$$\xleftarrow{1.6.5}$$