1.9 – Compositions of Matrix Transformations

Definitions: The **composition of T_B with T_A** is achieved by first applying the matrix transformation T_A to a vector and then applying the matrix transformation T_B to the image vector. We denote the composition of T_B with T_A by $T_B \circ T_A$ which is read "T_B circle T_A." This is also expressed as $(T_B \circ T_A)(\mathbf{x}) = T_B(T_A(\mathbf{x}))$.

Theorem 1.9.1 If $T_A : R^n \to R^k$ and $T_B : R^k \to R^m$ are matrix transformations, then $T_B \circ T_A$ is also a matrix transformation, and $T_B \circ T_A = T_{BA}$.

Pf: Let T_A, T_B be as described and $\vec{x} \in \mathbb{R}^n$.
Then $T_A(\vec{x}) = A\vec{x}$, $T_B(\vec{x}) = B\vec{x}$ by def of matrix transf.
$(T_B \circ T_A)(\vec{x}) = T_B(T_A(\vec{x}))$ by def of composition.
$= T_B(A\vec{x}) = B(A\vec{x}) = (BA)\vec{x}$ by associativity of matrix mult. $= T_{BA}(\vec{x})$.

#8 Find the standard matrix for the stated composition in R^2.
a. A rotation about the origin of 60°, followed by an orthogonal projection onto the x-axis, followed by a reflection about the line $y = x$.
b. An orthogonal projection onto the x-axis, followed by a rotation about the origin of 45°, followed by a reflection about the y-axis.
c. A rotation about the origin of 15°, followed by a rotation about the origin of 105°, followed by a rotation about the origin of 60°.

a) $R_{60°} = \begin{bmatrix} \cos 60° & -\sin 60° \\ \sin 60° & \cos 60° \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$T_B \circ T_A \circ R_{60^\circ} = BAR_\theta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ 0 & 0 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 0 & 0 \\ 1/2 & -\sqrt{3}/2 \end{bmatrix}}$$

Note $\left(T_B \circ T_A \circ R_{60^\circ}\right)\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1/2\, x_1 - \sqrt{3}/2\, x_2 \end{bmatrix}$

c) $R_{60^\circ} \circ R_{105^\circ} \circ R_{15^\circ} = R_{180^\circ} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

Bonus: Rotation α followed by Rotation β:

$$\begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} = \begin{bmatrix} \cos\beta\cos\alpha - \sin\beta\sin\alpha & -(\sin\alpha\cos\beta + \cos\alpha\sin\beta) \\ \sin\alpha\cos\beta + \cos\alpha\sin\beta & \cos\alpha\cos\beta - \sin\alpha\sin\beta \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha+\beta) & -\sin(\alpha+\beta) \\ \sin(\alpha+\beta) & \cos(\alpha+\beta) \end{bmatrix} = R_{\alpha+\beta}$$

#12 Let $T_1(x_1, x_2, x_3) = (4x_1, -2x_1 + x_2, -x_1 - 3x_2)$ and $T_2(x_1, x_2, x_3) = (x_1 + 2x_2, -x_3, 4x_1 - x_3)$. formulas

a. Find the standard matrices for T_1 and T_2.

b. Find the standard matrices for $T_2 \circ T_1$ and $T_1 \circ T_2$.

c. Use the matrices obtained in part (b) to find formulas for $T_1(T_2(x_1, x_2, x_3))$ and $T_2(T_1(x_1, x_2, x_3))$.

a) $[T_1] = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -3 & 0 \end{bmatrix}$, $[T_2] = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 4 & 0 & -1 \end{bmatrix}$

b) $[T_2 \circ T_1] = [T_2][T_1] = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 4 & 0 & -1 \end{bmatrix}\begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 \\ 1 & 3 & 0 \\ 17 & 3 & 0 \end{bmatrix}$

$$[T_1 \circ T_2] = [T_1][T_2] = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & -1 \\ 4 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 0 \\ -2 & -4 & -1 \\ -1 & -2 & 3 \end{bmatrix}$$

c)

$$T_1(T_2(\vec{x})) = (T_1 \circ T_2)(\vec{x}) = (4x_1 + 8x_2, -2x_1 - 4x_2 - x_3, -x_1 - 2x_2 + 3x_3)$$

$$T_2(T_1(\vec{x})) = (T_2 \circ T_1)(x_1, x_2, x_3) = (x_2, x_1 + 3x_2, 17x_1 + 3x_2)$$

Definition: If $T_A : R^n \to R^n$ is a matrix operator whose standard matrix A is invertible, then T_A is **invertible**, and the **inverse** of T_A is $T_A^{-1} = T_{A^{-1}}$.

#20 Determine whether the matrix operator $T : R^3 \to R^3$ defined by the equations is invertible; if so, find the standard matrix for the inverse operator, and find $T^{-1}(w_1, w_2, w_3)$.

a. $$\begin{aligned} w_1 &= x_1 - 2x_2 + 2x_3 \\ w_2 &= 2x_1 + x_2 + x_3 \\ w_3 &= x_1 + x_2 \end{aligned}$$

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 2 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 4 \\ 0 & 1 & 0 & -1 & 2 & -3 \\ 0 & 0 & 1 & -1 & 3 & -5 \end{array}\right]$$

$$T^{-1}(w_1, w_2, w_3)$$

$$= (w_1 - 2w_2 + 4w_3, -w_1 + 2w_2 - 3w_3, -w_1, 3w_2 - 5w_3)$$

$w_1 = x_1 - 3x_2 + 4x_3$

b. $w_2 = - x_1 + x_2 + x_3$

$w_3 = \quad - 2x_2 + 5x_3$

$$\left[\begin{array}{ccc|ccc} 1 & -3 & 4 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -2 & 5 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 + R_1$

$$\begin{array}{cccccc} -1 & 1 & 1 & 0 & 1 & 0 \\ 1 & -3 & 4 & 1 & 0 & 0 \\ \hline 0 & -2 & 5 & 1 & 1 & 0 \end{array}$$

$$\left[\begin{array}{ccc|ccc} 1 & -3 & 4 & 1 & 0 & 0 \\ 0 & -2 & 5 & 1 & 1 & 0 \\ 0 & -2 & 5 & 0 & 0 & 1 \end{array}\right]$$

This is not invertible.