• In-Person $\frac{dy}{dx} = y' = x^2 + y$ www.matthenes.com

- Syllabus, Schedule
- Fillable notes

2.2 - Separable Equations

• WebAssign

Definition: A **differential equation** (DE) is an equation containing the derivatives of one or more unknown functions (dependent variables) with respect to one or more independent variables.

Examples: $\frac{dy}{dx} + 5y = e^x$, $y'' - y' + 6y = 0$, and $\frac{dx}{dt} + \frac{dy}{dt} = 2x + y$ are ordinary differential equations (ODEs); $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ is an example of a partial differential equation (PDE). In this course, we will only work with ODEs.

Definition: The **order** of a DE is the order of the highest derivative in the equation.

Definition: A first-order **separable** DE has the form $\frac{dy}{dx} = g(x)h(y)$.

To solve, we separate variables and integrate:

$\frac{1}{h(y)}\,dy = g(x)\,dx$ OR $P(y)\,dy = g(x)\,dx$

Then $\int P(y)\,dy = \int g(x)\,dx$

Example: Solve the given differential equation by separation of variables.

$dy - (y-1)^2 dx = 0$

Note: This is a nonlinear DE because the equation contains a nonlinear function of y (the dependent variable).

$dy = (y-1)^2\,dx$

$\frac{1}{(y-1)^2}\,dy = dx$

$\int \frac{1}{(y-1)^2}\,dy = \int dx$

$y = 1 \Rightarrow dy = 0$ is a singular solution

$u = y - 1 \Rightarrow du = dy$

$$-\frac{1}{y-1} = x + C$$

$$\frac{1}{y-1} = -(x+C)$$

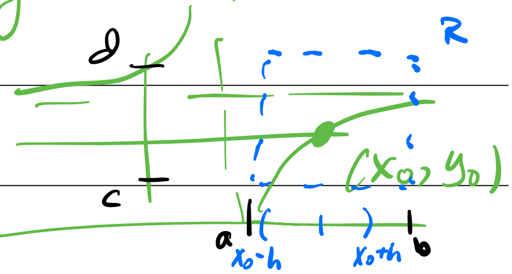
$$y - 1 = -\frac{1}{x+C}$$

$$y = 1 - \frac{1}{x+C}$$

One-parameter family of solutions

Suppose we know $C = -3$.

$$y = 1 - \frac{1}{x-3}$$



Definition: Any function ϕ, defined on an interval I and possessing at least n derivatives that are continuous on I, which when substituted into an nth-order ODE reduces the equation to an identity, is said to be a **solution** of the equation on the interval.

Example: $\sin 3x \; dx + 2y\cos^3 3x \; dy = 0$

$$2y\,dy = \int -\frac{\sin 3x}{\cos^3 3x}\,dx$$

$u = \cos 3x$

$du = -3\sin 3x\,dx$

$$y^2 = -\frac{1}{6}\cos^{-2} 3x + C$$

$$y^2 = -\frac{1}{6}\sec^2 3x + C$$

Example: $\dfrac{dN}{dt} + N = Nte^{t+2}$

Note: This is a linear DE because all instances of N (the dependent variable) are linear.

$$\frac{dN}{dt} = Nte^{t+2} - N = N(te^{t+2} - 1)$$

$$\frac{dN}{N} = (te^{t+2} - 1)\,dt$$

$u = t \quad dv = e^{t+2}dt$

$du = dt \quad v = e^{t+2}$

$$e^{\ln|N|} = e^{(te^{t+2} - e^{t+2} - t + c_1)}$$

$$|N| = e^{c_1} e^{(t-1)e^{t+2} - t}$$

$x^{2+3} = x^2 x^3$

$$c = \pm e^{c_1}$$

$$N(t) = c\, e^{(t-1)e^{t+2} - t}$$

Definition: Information such as $y(x_0) = y_0$ is called an **initial condition**. A DE, together with an initial condition is called an **initial-value problem** (IVP).

Isolate y

Example: Find an explicit solution of the given initial-value problem. Determine the exact interval I of definition by analytical methods. Use a graphing utility to plot the graph of the function.

$(2y - 2)\dfrac{dy}{dx} = 3x^2 + 4x + 2, \qquad y(1) = -2$

$$\int 2(y-1)\,dy = \int (3x^2 + 4x + 2)\,dx$$

$$(y-1)^2 = x^3 + 2x^2 + 2x + C$$

When $x = 1$, $y = -2$

$$9 = 5 + C \Rightarrow C = 4$$

$$(y-1)^2 = x^3 + 2x^2 + 2x + 4$$

$y-1 = \pm\sqrt{x^3+2x^2+2x+4}$

$y = 1 \pm \sqrt{x^3+2x^2+2x+4}$

$y = 1 - \sqrt{x^3+2x^2+2x+4}$

derivative must be defined

Interval: $x^3+2x^2+2x+4 > 0$

$x^2(x+2)+2(x+2)>0 \Rightarrow (x^2+2)(x+2)>0$

so $(-2,\infty)$ is the interval of solution

Theorem: Existence of a Unique Solution

Consider the first-order initial-value problem

Solve $\dfrac{dy}{dx} = f(x, y)$, subject to $y(x_0) = y_0$

Let R be a rectangular region in the xy-plane defined by $a \le x \le b,\ c \le y \le d$ that contains the point (x_0, y_0) in its interior. If $f(x, y)$ and $\dfrac{\partial f}{\partial y}$ are continuous on R, then there exists some interval $I_0\ :\ (x_0-h, x_0+h),\ h > 0$, contained in $[a,\ b]$, and a unique function $y(x)$, defined on I_0, that is a solution of the initial-value problem.

We will consider higher-order DEs later in the course.

Example: Find an explicit solution of the given initial-value problem. Give the solution using a definite integral.

$$\frac{dy}{dx} = y^2 \sin x^2, \qquad y(-2) = \frac{1}{3}$$

no antiderivative

$\int y^{-2}\,dy = \int \sin(x^2)\,dx$ $\qquad$ $\int e^{-x^2}\,dx$

create definite integrals

$$\int_{-2}^{x} y^{-2}(t)\, dy = \int_{-2}^{x} \sin(t^2)\, dt$$

t is a dummy variable

$$-\frac{1}{y(t)}\Big|_{-2}^{x} = \int_{-2}^{x} \sin(t^2)\, dt$$

$$-\frac{1}{y(x)} + \frac{1}{y(-2)} = \int_{-2}^{x} \sin(t^2)\, dt$$

$$-\frac{1}{y(x)} + 3 = \int_{-2}^{x} \sin(t^2)\, dt$$

$$y(x) = \left[3 - \int_{-2}^{x} \sin(t^2)\, dt\right]^{-1}$$