2.3 - Linear Equations

yy' is nonlinear

Definition: A first-order differential equation of the form
$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$ is a **linear equation** in y (the dependent variable).

input function / forcing function

This is put into **standard form** by dividing both sides by $a_1(y)$ to obtain

$$\frac{dy}{dx} + P(x)y = f(x)$$

Example: Consider the DE $3\frac{dy}{dx} + 12y = 4$. In standard form, this is
$\frac{dy}{dx} + 4y = \frac{4}{3}$.

Wha?

If we multiply both sides by e^{4x},

we get $\underbrace{e^{4x}\frac{dy}{dx} + 4e^{4x}y}$ $= \frac{4}{3}e^{4x}$

Aside: differentiate $e^{4x}y$ with respect to x:

2nd Aside: Implicit differentiation

$\underbrace{x^2y^3} + 4xy^2 = 5$

$\frac{d}{dx}(x^2y^3) = 2xy^3 + 3x^2y^2\frac{dy}{dx}$

$(uv)' = u'v + uv'$

$\frac{d}{dx}(e^{4x}y) = \underbrace{e^{4x}\frac{dy}{dx} + 4e^{4x}y}$

$$\int \frac{d}{dx}(e^{4x}y)\,dx = \int \frac{4}{3}e^{4x}\,dx$$

$$e^{4x}y = \frac{1}{3}e^{4x} + C$$

$$y = \frac{1}{3} + Ce^{-4x}$$

e^{4x} is called an integrating factor.

Where did e^{4x} come from?

For the eqn. in std. form

$$\frac{dy}{dx} + P(x)y = f(x), \quad (*)$$

mu

Suppose we have some function $\mu(x)$ such that when we multiply both sides of $(*)$ by $\mu(x)$, we get

$$\mu(x)\frac{dy}{dx} + \mu(x)P(x)y = \mu(x)f(x)$$

If it happens that $\boxed{\frac{d\mu}{dx} = \mu P}$, then the LHS becomes

$$\mu \frac{dy}{dx} + \frac{d\mu}{dx} y = \mu f(x)$$

Then we have $\frac{d}{dx}(\mu y) = \mu f(x)$ ...

To find μ, solve the separable equation

$$\frac{d\mu}{dx} = \mu P \implies \frac{d\mu}{\mu} = P(x)\,dx$$

$$\ln|\mu| = \int P(x)\,dx \implies \boxed{\mu(x) = e^{\int P(x)\,dx}}$$

Example: Find the general solution of the given differential equation. Give the largest interval I over which the general solution is defined. Determine whether there are any transient terms in the general solution.

Not necessary when finding/using an integrating factor

$y' + 2xy = x^3$

$$P(x) = 2x \implies \mu = e^{\int 2x\,dx} = e^{x^2 + C}$$

$$e^{x^2} y' + 2x e^{x^2} y = x^3 e^{x^2}$$

$$\mu y' + \mu' y$$

$$\frac{d}{dx}\left(e^{x^2} y\right) = x^3 e^{x^2}$$

$$e^{x^2} y = \int x^3 e^{x^2}\,dx \qquad u = x^2 \quad dv = x e^{x^2}\,dx$$

you can verify that we get

$$e^{x^2}y = \frac{1}{2}x^2e^{x^2} - \int xe^{x^2}dx$$

$$e^{x^2}y = \frac{1}{2}x^2e^{x^2} - \frac{1}{2}e^{x^2} + C$$

$$y = \underbrace{\frac{1}{2}x^2 - \frac{1}{2}}_{\text{steady state solution}} + \underset{\uparrow \text{ transient term}}{Ce^{-x^2}}$$

Interval: $(-\infty, \infty)$

Example: $(1+x)\frac{dy}{dx} - xy = x + x^2$

Std form $\quad \frac{dy}{dx} - \frac{x}{1+x}y = x$

$$P(x) = -\frac{x}{1+x} \Rightarrow \mu = e^{\int -\frac{x}{1+x}dx}$$

$$-\int \frac{x+1-1}{1+x}dx = -\int\left(1 - \frac{1}{x+1}\right)dx$$

$$\mu = e^{\ln|x+1| - x} = e^{\ln|x+1|}e^{-x} = (x+1)e^{-x}$$

$$\int \frac{d}{dx}\left[(x+1)e^{-x}y\right]dx = \int (x^2+x)e^{-x}dx$$

$u = x^2 + x \qquad dv = e^{-x}dx$

$du = (2x+1)dx \qquad v = -e^{-x}$

$$(x+1)e^{-x}y = -(x^2+x)e^{-x} + \int(2x+1)e^{-x}dx$$

$u = 2x+1 \quad dv = e^{-x}dx$

$du = 2dx \quad v = -e^{-x}$

$$(x+1)e^{-x}y = -(x^2+x)e^{-x} \underbrace{-(2x+1)}_{-(2x+3)}e^{-x} \underbrace{-2}e^{-x} + C$$

$$y = -x - \frac{2x+3}{x+1} + \frac{Ce^x}{x+1}$$

No transient terms.

solution is defined on $(-\infty, -1) \cup (-1, \infty)$

Interval of solution: $(-1, \infty)$

Example: $xy' + (1+x)y = e^{-x}\sin 2x$

$$y' + \left(\frac{1}{x}+1\right)y = \frac{\sin 2x}{xe^{x}}$$

$$\mu = e^{\int\left(\frac{1}{x}+1\right)dx}$$

$$\mu = xe^{x}$$

$$\frac{d}{dx}\left[xe^{x}y\right] = \sin 2x$$

$$xe^{x}y = -\frac{1}{2}\cos 2x + C$$

$$y = -\frac{1}{2x}e^{-x}\cos 2x + \frac{c}{x}e^{-x}$$

Interval of solution $(0,\infty)$

It's all transient

Indep: t
Dep: P

Example: $\dfrac{dP}{dt} + 2tP = P + 4t - 2$

$\mu P' + \mu' P$

$$\mu \frac{dP}{dt} + \mu(2t-1)P = \mu(4t-2)$$

$$\mu = e^{\int (2t-1)\,dt} = e^{t^2-t}$$

$$e^{t^2-t}P' + (2t-1)e^{t^2-t}P = (4t-2)e^{t^2-t}$$

$$\frac{d}{dt}\left[e^{t^2-t}P\right] = (4t-2)e^{t^2-t}$$

t

$u = t^2 - t \Rightarrow du = (2t-1)dt$

$$\int \frac{d}{dt}\left[e^{t^2-t}P\right]dt = \int 2(2t-1)e^{t^2-t}\,dt$$

$$e^{t^2-t}P = 2e^{t^2-t} + C$$

$$P(t) = 2 + Ce^{t-t^2}$$

Transient term: Ce^{t-t^2}

Interval: $(-\infty, \infty)$

Example: $(x^2-1)\dfrac{dy}{dx} + 2y = (x+1)^2$

Linear if std form is $\dfrac{dy}{dx} + P(x)y = f(x)$

$$\frac{dy}{dx} + \frac{2}{x^2-1}y = \frac{x+1}{x-1}$$

$$P(x) = \frac{2}{x^2-1} \Rightarrow \int P(x)\,dx = \int \frac{2}{x^2-1}\,dx$$

$$\frac{2}{(x+1)(x-1)} = \frac{A}{x+1} + \frac{B}{x-1} \Rightarrow 2 = A(x-1) + B(x+1)$$

$$x=-1 \Rightarrow A=-1$$

$$x=1 \Rightarrow B=1$$

$$\mu = e^{\int\left(\frac{1}{x-1} - \frac{1}{x+1}\right)dx} = \frac{x-1}{x+1}$$

$$\frac{x-1}{x+1}\frac{dy}{dx} + \underbrace{\frac{x-1}{x+1}\frac{2}{x^2-1}}_{\mu'}y = \frac{x-1}{x+1}\frac{x+1}{x-1}$$

$$\frac{d}{dx}[\mu y] = \mu y' + \mu' y$$

$$\int \frac{d}{dx}\left[\frac{x-1}{x+1}y\right]dx = \int 1\,dx$$

$$\frac{x-1}{x+1}y = x + C \Rightarrow y = \frac{x^2+x}{x-1} + \frac{C(x+1)}{x-1}$$

No transient terms

Interval : $(1, \infty$

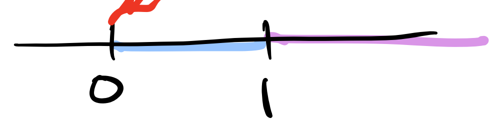
Desmos
Geogebra

Example: Solve the given initial-value problem. Use a graphing utility to graph the continuous function $y(x)$.

$$(1+x^2)\frac{dy}{dx} + 2xy = f(x), \qquad y(0) = 0, \text{ where}$$

$x=0$

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ -x & x \ge 1 \end{cases}$$



$0 \le x < 1$

$x \ge 1$

$$(1+x^2)\frac{dy}{dx} + 2xy = x \qquad (1+x^2)\frac{dy}{dx} + 2xy = -x$$

$$\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{x}{1+x^2} \qquad \frac{dy}{dx} + \frac{2x}{x^2+1}y = -\frac{x}{1+x^2}$$

$$\mu = e^{\int \frac{2x}{1+x^2}dx} = 1+x^2$$

$$\frac{d}{dx}\left[(1+x^2)y\right] = x \qquad \frac{d}{dx}\left[(1+x^2)y\right] = -x$$

$$(1+x^2)y = \frac{1}{2}x^2 + C_1 \qquad (1+x^2)y = -\frac{1}{2}x^2 + C_2$$

$$y = \frac{x^2}{2(1+x^2)} + \frac{C_1}{1+x^2} \qquad y = -\frac{x^2}{2(1+x^2)} + \frac{C_2}{1+x^2}$$

$y(0)=0$ $\quad C_1 = 0$

$0 \le x < 1$

$$y = \frac{x^2}{2(1+x^2)}$$

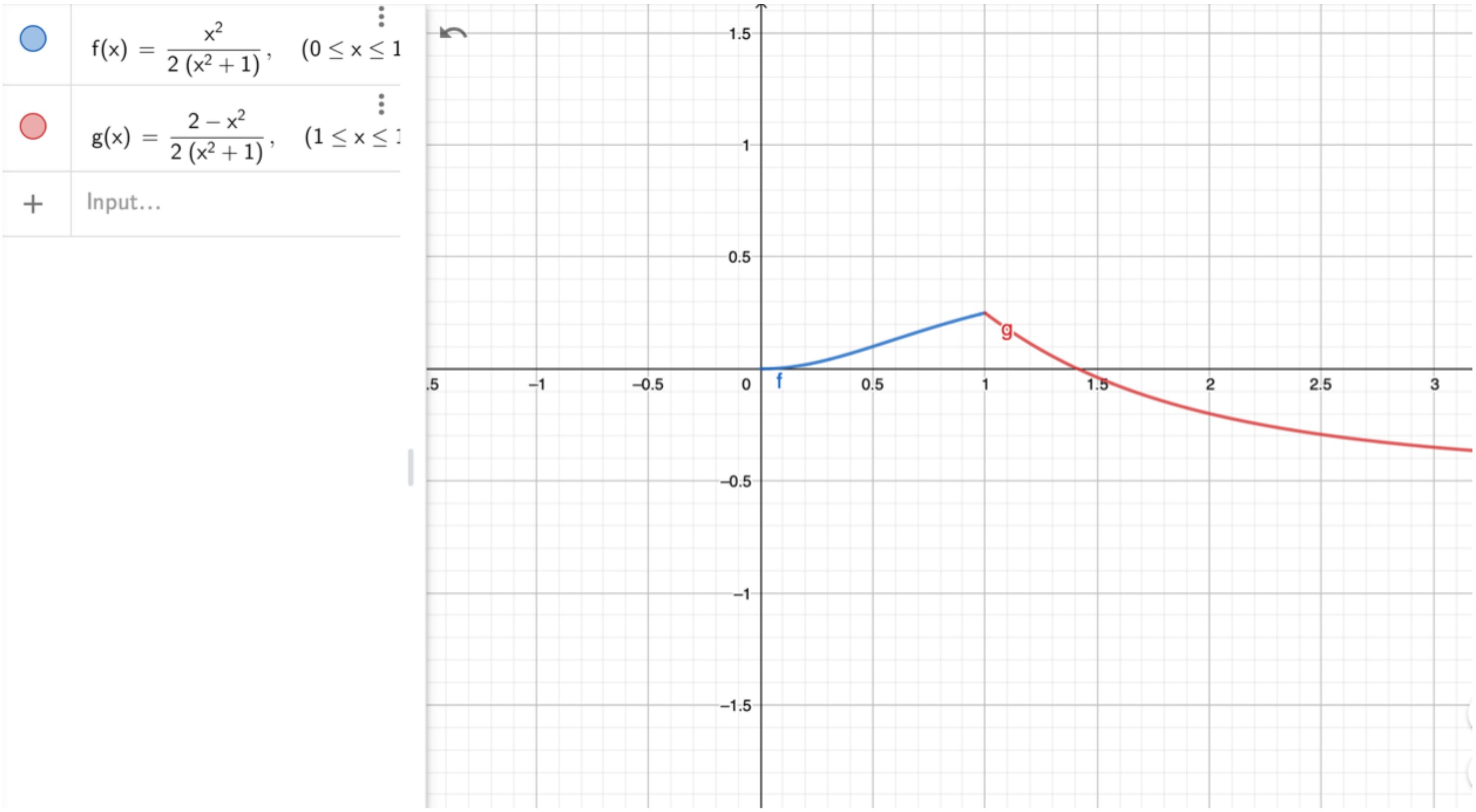
$$y = \begin{cases} \frac{x^2}{2(1+x^2)}, & 0 \le x < 1 \\ -\frac{x^2}{2(1+x^2)} + \frac{C_2}{1+x^2}, & x \ge 1 \end{cases}$$

y continuous $\Rightarrow$ y cont. at $x=1$

$$\Rightarrow \lim_{x \to 1^-} y = \lim_{x \to 1^+} y$$

$$\frac{1}{4} = -\frac{1}{4} + \frac{C_2}{2} \Rightarrow C_2 = 1$$

$$y = \begin{cases} \frac{x^2}{2(1+x^2)}, & 0 \le x < 1 \\ \frac{2-x^2}{2(1+x^2)}, & x \ge 1 \end{cases}$$



$f(x) = \frac{x^2}{2(x^2+1)}, \quad (0 \le x \le 1$
$g(x) = \frac{2-x^2}{2(x^2+1)}, \quad (1 \le x \le$
Input...
1.5
1
0.5
0
-0.5
-1
-1.5
.5
-1
-0.5
0.5
1
1.5
2
2.5
3
f
g