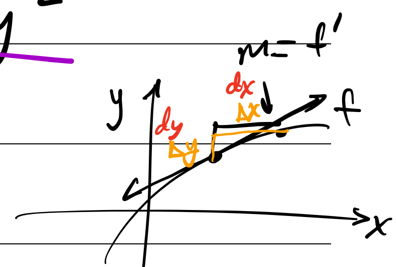
2.4 - Exact Equations

A partial derivative is an ordinary derivative holding all but one variable constant.

Computation & Notation: $f(x,y) = x^2y^3$

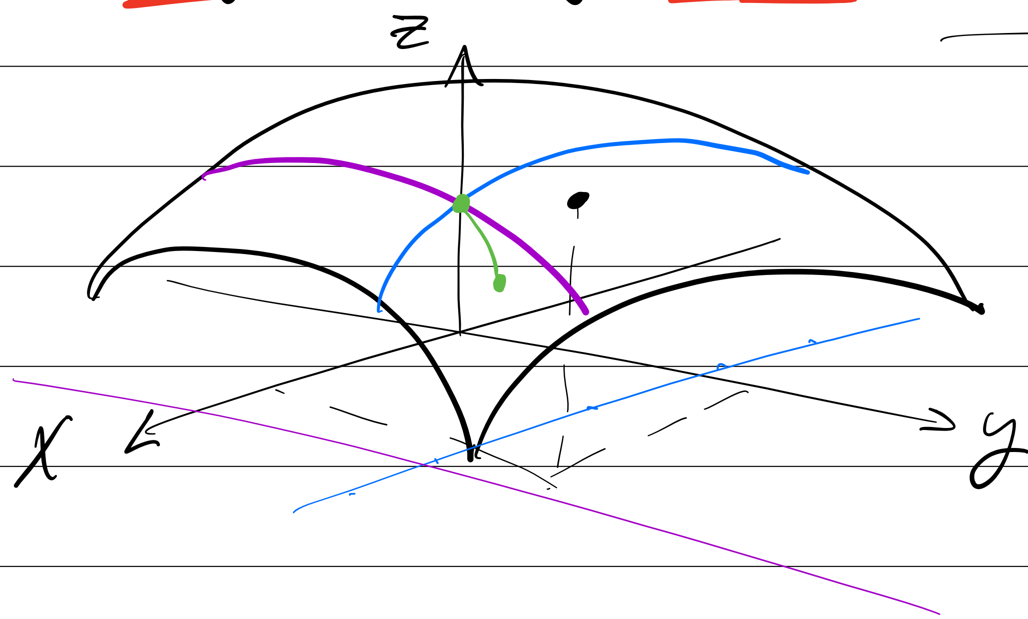
$f_x = \frac{\partial f}{\partial x} = 2xy^3$ $\qquad$ $f_y = \frac{\partial f}{\partial y} = 3x^2y^2$

$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = 6xy^2$ $\qquad$ $f_{yx} = 6xy^2$



$\frac{dy}{dx} = f'(x)$

$dy = f'(x)\,dx$

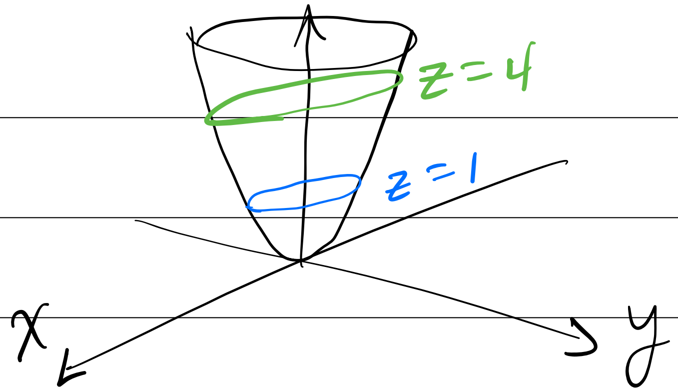


the full differential of f is

$$f_x\,dx + f_y\,dy$$

Consider $f(x,y) = C$ as a level curve to a surface. Ex: If $f(x,y) = x^2+y^2$

$z = f(x,y) = C$ becomes $x^2+y^2 = C$



$x^2+y^2=C$ is a level curve of this surface

Considering the level curve $f(x,y)=c$, the (full) exact differential is

$$f_x\,dx + f_y\,dy = 0$$

For our work, we will consider $M\,dx+N\,dy=0$, the solution to which is $\boxed{f(x,y)=C}$

$$\begin{tikzcd} & f(x,y) \arrow[dl, "\frac{\partial}{\partial x}"'] \arrow[dr, "\frac{\partial}{\partial y}"] & \\ M & & N \end{tikzcd}$$

$f_x \overset{?}{=}$ $\boxed{M \quad \text{starting point} \quad N}$ $\overset{?}{=} f_y$

Test for exactness

$$f_{xy} = M_y \overset{?}{=} N_x = f_{yx}$$

Example: Determine whether the given differential equation is exact. If it is exact, solve it.

$(\sin y - y \sin x)\, dx + (\cos x + x \cos y - y)\, dy = 0$

$M = \sin y - y \sin x \qquad N = \cos x + x\cos y - y$

$M_y = \cos y - \sin x \qquad N_x = -\sin x + \cos y$ ✓ exact

$f(x,y) = \int M\,dx = \int (\sin y - y\sin x)\,dx$ ← const WRT x

$= x\sin y + y\cos x + h(y)$

$N = f_y$ so $\cancel{\cos x} + \cancel{x\cos y} - y = \cancel{x\cos y} + \cancel{\cos x} + h'(y)$

$-y = h'(y) \Rightarrow -\frac{1}{2}y^2 + k = h(y)$

$c = -k$

$$\boxed{\underbrace{x\sin y + y\cos x - \tfrac{1}{2}y^2}_{f(x,y)} = C}$$

Example: $\left(1+\ln x+\frac{y}{x}\right)\,dx=(1-\ln x)\,dy$ Form: $M\,dx+N\,dy=0$

$M=1+\ln x+\frac{y}{x}$, $N=\ln x-1$

$M_y=\frac{1}{x}$ $N_x=\frac{1}{x}$ ✓

$f(x,y)=\int M\,dx=\int\left(1+\ln x+\frac{y}{x}\right)dx$ Eww

$f(x,y)=\int N\,dy=\int(\ln x-1)\,dy=y\ln x-y+g(x)$

$M=f_x$ so $1+\ln x+\frac{y}{x}=\frac{y}{x}+g'(x)$

$g'(x)=1+\ln x\Rightarrow g(x)=\int(1+\ln x)\,dx$

$\int \ln x\,dx$ $u=\ln x$ $dv=dx$

$du=\frac{1}{x}dx$ $v=x$

$g(x)=x+x\ln x-x+k=x\ln x+k$

Form: $f(x,y)=C$

$$\boxed{y\ln x-y+x\ln x=C}$$

Note: The form of solution is not $y=$ stuff or $f(x,y)=$ stuff.

Example: Solve the given initial-value problem.

$(e^x + y)\ dx + (2 + x + ye^y)\ dy = 0 \qquad y(0) = 1$

$M_y = 1 \qquad N_x = 1$ ✓ exact

① $f(x,y) = \int M\,dx = \int (e^x + y)\,dx = e^x + xy + h(y)$

② $f(x,y) = \int N\,dy = \int (2 + x + ye^y)\,dy$

$= 2y + xy + \int ye^y\,dy$

$\int ye^y\,dy \qquad u = y \quad dv = e^y\,dy$

$du = dy \quad v = e^y$

$ye^y - e^y$

$\int N\,dy = 2y + xy + ye^y - e^y + g(x)$

$2y + xy + ye^y - e^y + e^x = C$

check: $f_x = y + e^x \qquad f_y = 2 + x + ye^y + e^y - e^y$

$= 2 + x + ye^y$

$f_x\,dx + f_y\,dy = 0$

is $(y + e^x)dx + (2 + x + ye^y) = 0$ ✓

$y(0) = 1$ (When $x = 0$, $y = 1$) $\quad 2 + 0 + e - e + 1 = C \Rightarrow C = 3$

$\boxed{2y + xy + ye^y - e^y + e^x = 3}$

Example: Solve the given differential equation by finding an appropriate integrating factor.

$$\cos x \, dx + \left(1 + \frac{2}{y}\right) \sin x \, dy = 0$$

$M_y = 0$ $\quad N_x = (1 + \frac{2}{y})\cos x \quad$ & not exact

the integrating factor is either

$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}$$

OR $\quad \mu(y) = e^{\int \frac{N_x - M_y}{M} dy}$

$$\mu(x) = e^{\int \frac{-\cancel{(1+\frac{2}{y})}\cos x}{\cancel{(1+2/y)}\sin x} dx} = e^{\ln|\sin x|^{-1}}$$

$\mu(x) = \csc x$

$$\cot x \, dx + (1 + \tfrac{2}{y}) \, dy = 0$$ this is exact

$f(x,y) = \int \cot x \, dx = \ln|\sin x| + h(y)$

$\int \frac{\cos x}{\sin x} dx = \int \frac{du}{u} = \ln|\sin x|$

$u = \sin x$

$f_y = N \Rightarrow h'(y) = 1 + \frac{2}{y}$

$h(y) = y + \ln y^2 + k$

$$\boxed{\ln|\sin x| + y + \ln y^2 = C}$$