2.5 - Solutions by Substitutions

We will use substitutions to solve DEs (and IVPs) of the following types.

$\left(y^2 + xy\right)\ dx + x^2\ dy = 0$

An equation of the form $M(x, y)\,dx + N(x, y)\,dy = 0$ is a homogeneous equation, if the coefficient functions possess the property $f(tx, ty) = t^{\alpha} f(x, y)$ (with the same α for both functions).

$M(x,y) = y^2 + xy \Rightarrow M(tx, ty) = (ty)^2 + (tx)(ty) = t^2(y^2 + xy)$

Substitution:

$y = ux$ OR $x = vy$

$dy = x\,du + u\,dx$

$\frac{dy}{dx} - y = e^x y^2$

The DE $\frac{dy}{dx} + P(x)y = f(x)y^n$ is called Bernoulli's equation. $n \neq 1$

Substitution:

$u = y^{1-n}$

The equation $\frac{dy}{dx} = f(Ax + By + C)$ can be converted to a separable equation using the substitution $u = Ax + By + C$.

$$u = Ax + By + C \Rightarrow \frac{du}{dx} = A + B\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{B}\left(\frac{du}{dx} - A\right)$$

$$\frac{dy}{dx} = \frac{3x + 2y}{3x + 2y + 2}$$

Example: Solve the given differential by using an appropriate substitution.

$$\left(y^2 + xy\right)\ dx + x^2\ dy = 0$$

use either $y = ux$ or $x = vy$

$dy = x\,du + u\,dx$ $\qquad$ $dx = y\,dv + v\,dy$

$$(u^2x^2 + x\,ux)dx + x^2(x\,du + u\,dx) = 0$$

$$u^2\cancel{x^2}dx + \cancel{x^2}u\,dx + \cancel{x^3}du + \cancel{x^2}u\,dx = 0$$

Divide out x $\qquad (u^2 + 2u)dx + x\,du = 0$

$$\int \frac{du}{u^2 + 2u} + \int \frac{dx}{x} = \int 0\,dx$$

PFD

$$\int \left(\frac{1/2}{u} - \frac{1/2}{u+2}\right)du + \int \frac{dx}{x} = \int 0\,dx$$

OR $\int (\ )\,du = -\int \frac{1}{x}\,dx \Rightarrow \ln|x| +$

$$\ln\sqrt{\frac{u}{u+2}} + \ln|x| = \ln C_1$$

C_1^2

$$\sqrt{\frac{u}{u+2}}\,x = C_1 \Rightarrow \frac{u}{u+2} = \frac{c}{x^2}$$

$$u = \frac{y}{x}, \text{ so } \qquad \frac{\cancel{x}\ y/x}{\cancel{x}\ y/x + 2} = \frac{c}{x^2}$$

$$\frac{y}{y+2x} = \frac{c}{x^2}$$

$$x^2 y = c(2x+y)$$

Example: $\frac{dy}{dx} - y = e^x y^2$ $\quad n=2$

$$u = y^{1-2} \Rightarrow u = y^{-1} \Rightarrow y = u^{-1}$$

$$\frac{dy}{dx} = -u^{-2}\frac{du}{dx}$$

$$-u^{-2}\frac{du}{dx} - \frac{1}{u} = e^x u^{-2}$$

$$\frac{du}{dx} + u = -e^x$$

$P(x)=1 \Rightarrow \mu(x) = e^x$

$$\frac{d}{dx}(e^x u) = -e^{2x} \Rightarrow e^x u = -\frac{1}{2}e^{2x} + C_1$$

$$u = -\frac{1}{2}e^x + C_1 e^{-x}$$

$C = 2C_1$

$$u = -\frac{e^x}{2} + \frac{C_1}{e^x} = \frac{-e^{2x}+C}{2e^x}$$

$$\frac{1}{y} = \frac{C-e^{2x}}{2e^x}$$

$$y = \frac{2e^x}{C-e^{2x}}$$

Example: $3\left(1+t^2\right)\frac{dy}{dt} = 2ty\left(y^3-1\right)$ $\quad$ Want: $\frac{dy}{dt} + P(t)y = f(t)y^n$

$$3(1+t^2)\frac{dy}{dt} + 2ty = 2ty^4$$

$$\frac{dy}{dt} + \frac{2t}{3(1+t^2)} y = \frac{2t}{3(1+t^2)} y^4$$

Bernoulli $n=4$

$$u = y^{1-4} \Rightarrow u = y^{-3} \Rightarrow y = u^{-1/3}$$

$$u^{-1/3} = (y^{-3})^{-1/3}$$

$$\frac{dy}{dt} = -\frac{1}{3} u^{-4/3} \frac{du}{dt}, \quad y^4 = u^{-4/3}$$

$$-\frac{1}{3} u^{-4/3} \frac{du}{dt} + \frac{2t}{3(1+t^2)} u^{-1/3} = \frac{2t}{3(1+t^2)} u^{-4/3}$$

$$\frac{du}{dt} - \frac{2t}{1+t^2} u = -\frac{2t}{1+t^2}$$

$$\mu = e^{\int -\frac{2t}{1+t^2} dt} = e^{-\ln(1+t^2)} = \frac{1}{1+t^2}$$

$$\frac{d}{dt}\left(\frac{1}{1+t^2} u\right) = -\frac{2t}{(1+t^2)^2}$$

$$\frac{1}{1+t^2} u = \frac{1}{1+t^2} + C$$

$$u = 1 + C(1+t^2)$$

$$y^{-3} = 1 + C(1+t^2)$$ ← This is fine

Explicit: $y = [1 + C(1+t^2)]^{-1/3}$

Example: Solve the given initial-value problem.

$$\frac{dy}{dx} = \frac{3x+2y}{3x+2y+2}, \qquad y(-1) = -1$$

$u = 3x+2y$

$\frac{du}{dx} = 3 + 2\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{2}\left(\frac{du}{dx} - 3\right)$

$\frac{1}{2}\left(\frac{du}{dx} - 3\right) = \frac{u}{u+2}$

$\frac{du}{dx} = \frac{2u}{u+2} + 3\frac{u+2}{u+2}$

$\frac{du}{dx} = \frac{5u+6}{u+2}$

$\frac{u+2}{5u+6}\,du = dx$

OR

$u = 3x+2y+2$

$\frac{du}{dx} = 3+2\frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{2}\left(\frac{du}{dx} - 3\right)$

$\frac{1}{2}\left(\frac{du}{dx} - 3\right) = \frac{u-2}{u}$

$\frac{du}{dx} = \frac{2u-4}{u} + 3$

$\frac{du}{dx} = \frac{5u-4}{u}$

$\frac{u}{5u-4}\,du = dx$

$$\begin{array}{r|l} & \frac{1}{5} + \frac{4/5}{5u-4} \\ \hline 5u-4 & u \\ & \underline{-u + \frac{4}{5}} \end{array}$$

$\int\left(\frac{1}{5} + \frac{4/5}{5u-4}\right)du = dx$

$\frac{1}{5}u + \frac{4}{25}\ln|5u-4| = x + C_1$

$\cancel{\frac{1}{5}}(3x+2y+2) + \frac{4}{\cancel{25}\,5}\ln|15x+10y+6| = 5x + \cancel{C}\,C_2$

$-2x+2y + \frac{4}{5}\ln|15x+10y+6| = C$

$y(-1) = -1$ When $x = -1$, $y = -1$

$$C = \frac{4}{5}\ln|-19|$$

$$2y - 2x + \frac{4}{5}\ln|15x + 10y + 6| = \frac{4}{5}\ln 19$$