Exam 1: 2nd half of 2.1, 2.2 – 2.5, 3.1, 3.2

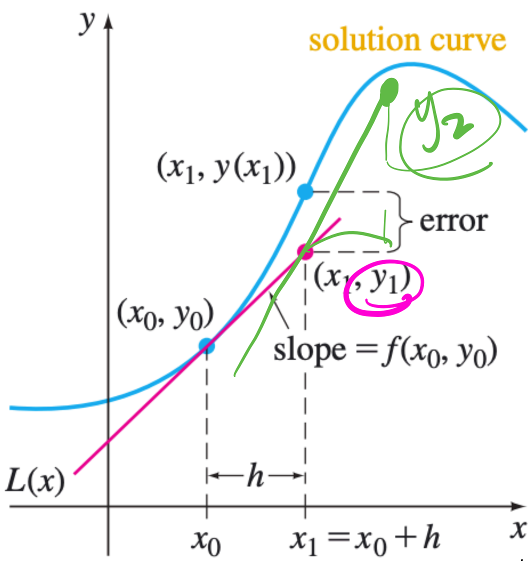
2.6 - A Numerical Method (Euler's Method)

$y - y_1 = m(x - x_1)$

Hidden Figures

Recall the equation of the tangent line of the solution curve for the differential equation $y' = f(x, y)$, $y(x_0) = y_0$ is $y = y_0 + f(x_0, y_0)(x - x_0)$.

Definition: This is known as the **linearization of f at** (x_0, y_0).



If we consider (x_1, y_1) with x-value $x_1 = x_0 + h$ (a small distance to the right of x_0), we have $y_1 = y_0 + h f(x_0, y_0)$

(x_1, y_1) is an approximation to $(x_1, y(x_1))$, which is on the solution curve.

Using (x_1, y_1) and the slope given by $y' = f(x, y)$ at that point, we get $y_2 = y_1 + h f(x_1, y_1)$ and (x_2, y_2), where $x_2 = x_0 + 2h = x_1 + h$

This process continues, and we can generalize the results:

$$x_{n+1} = x_n + h$$
$$y_{n+1} = y_n + hf(x_n, y_n),\ \ n = 0, 1, 2, \ldots$$

Example: Use Euler's method to obtain a four-decimal approximation of the indicated value. Carry out the recursion by hand, first using $h = 0.1$ and then using $h = 0.05$.

$y' = x + y^2, \qquad y(0) = 0; \qquad y(0.2)$

$h=0.1 \qquad x_0 = 0, \quad y_0 = 0$

$x_1 = 0.1, \quad y(0.1) \approx y_1 = 0 + 0.1(0 + 0^2) = 0$

$x_2 = 0.2, \quad y(0.2) \approx y_2 = 0 + 0.1(0.1 + 0^2)$

$y(0.2) \approx 0.01$

$h=0.05 \qquad x_0 = 0, \quad y_0 = 0$

$x_1 = 0.05, \quad y(0.05) = 0 + 0.05(0 + 0^2) = 0$

$x_2 = 0.1, \quad y(0.1) = 0 + 0.05(0.05 + 0^2)$

$= 0.0025$

$x_3 = 0.15, \quad y(0.15) \approx 0.0025 + 0.05(0.1 + 0.0025^2)$

$y(0.15) \approx 0.0075$

$x_4 = 0.2, \quad y(0.2) \approx 0.0075 + 0.05(0.15 + 0.0075^2)$

$y(0.2) \approx 0.0150$

Example: Use Euler's method to obtain a four-decimal approximation of the indicated value. First use $h = 0.1$ and then use $h = 0.05$. Find an explicit solution for each initial-value problem and then construct a table displaying estimated value, actual value, absolute error, and percentage relative error.

$$y' = 2xy, \qquad y(1) = 1; \qquad y(1.5)$$

h	x_n	y_n	f(x_n, y_n)
0.1	1	1	2
	1.1	1.2	2.64
	1.2	1.464	3.5136
	1.3	1.81536	4.719936
	1.4	2.2873536	6.4045901
	1.5	2.9278126	8.7834378

x_n	y_n	Actual value	Abs. error	% Rel. error
1	1	1	0	0.00%
1.1	1.2	1.2336781	0.0336781	2.73%
1.2	1.464	1.5527072	0.0887072	5.71%
1.3	1.81536	1.9937155	0.1783555	8.95%
1.4	2.2873536	2.6116965	0.3243429	12.42%
1.5	2.9278126	3.490343	0.5625303	16.12%

Analytic solution

$\frac{y'}{y} = 2x\,dx$ separable

$\ln|y| = x^2 + C$

$y(1) = 1 \Rightarrow 0 = 1 + C$

$C = -1 \Rightarrow y = e^{x^2 - 1}$

h	x_n	y_n	f(x_n, y_n)
0.05	1	1	2
	1.05	1.1	2.31
	1.1	1.2155	2.6741
	1.15	1.349205	3.1031715
	1.2	1.5043636	3.6104726
	1.25	1.6848872	4.212218
	1.3	1.8954981	4.9282951
	1.35	2.1419129	5.7831647
	1.4	2.4310711	6.8069991
	1.45	2.771421	8.037121
	1.5	3.1732771	9.5198313

x_n	y_n	Actual value	Abs. error	% Rel. error
1	1	1	0	0.00%
1.05	1.1	1.1079373	0.0079373	0.72%
1.1	1.2155	1.2336781	0.0181781	1.47%
1.15	1.349205	1.3805749	0.0313699	2.27%
1.2	1.5043636	1.5527072	0.0483436	3.11%
1.25	1.6848872	1.7550547	0.0701675	4.00%
1.3	1.8954981	1.9937155	0.0982174	4.93%
1.35	2.1419129	2.2761832	0.1342703	5.90%
1.4	2.4310711	2.6116965	0.1806254	6.92%
1.45	2.771421	3.0116858	0.2402648	7.98%
1.5	3.1732771	3.490343	0.3170659	9.08%

$y' - 2xy = 0$ linear

$\mu = e^{-x^2}$

$\frac{d}{dx}(e^{-x^2} y) = 0$

$e^{-x^2} y = C \Rightarrow y = Ce^{x^2}$

$y(1) = 1 \quad 1 = Ce$

$C = e^{-1}$

$y = e^{x^2} e^{-1} \Rightarrow y = e^{x^2 - 1}$