3.1 - Linear Models

Various situations are modeled by differential equations.

initial pop, amt

some populations, radioactive decay, drug concentrations

- Growth and decay: $\dfrac{dx}{dt} = kx, \qquad x(t_0) = x_0$

$\dfrac{dx/dt}{x} = k$ constant relative growth/decay rate

$k>0$ $k<0$

k is the growth/decay rate as a decimal

temp

- Newton's law of heating and cooling: $\dfrac{dT}{dt} = k(T - T_m), \qquad T(t_0) = T_0$

temp of medium

- Mixtures: $\dfrac{dA}{dt} = \text{rate in} - \text{rate out}, \qquad A(0) = A_0$
- LR-series electrical circuit: $L\dfrac{di}{dt} + Ri = E(t)$

L - inductance (henries - h) E - voltage

R - resistance (ohms - Ω) (volts - V)

i - current (amperes - A)

- RC-series electrical circuit: $R\dfrac{dq}{dt} + \dfrac{1}{C}q = E(t)$

$i = \dfrac{dq}{dt}$

C - capacitance (farads f)

q - charge (coulombs C)

Example: A population of bacteria in a culture grows at a rate proportional to the number of bacteria present at time t. After 3 hours it is observed that 400 bacteria are present. After 10 hours 2000 bacteria are present. What was the initial number of bacteria?

$$\frac{dP}{dt} = kP \Rightarrow \frac{dP}{dt} - kP = 0 \Rightarrow \mu = e^{-kt}$$

$$\frac{d}{dt}\left(e^{-kt}P\right) = 0 \Rightarrow e^{-kt}P = C$$

$$P = Ce^{kt}$$

Pop at time $t=0$ is initial population, P_0

Solution to model: $\boxed{P(t) = P_0 e^{kt}}$

$P(3) = 400$, $P(10) = 2000$

$$\frac{2000 = P_0 e^{10k}}{400 = P_0 e^{3k}}$$

$$5 = e^{7k} \Rightarrow k = \frac{1}{7}\ln 5$$

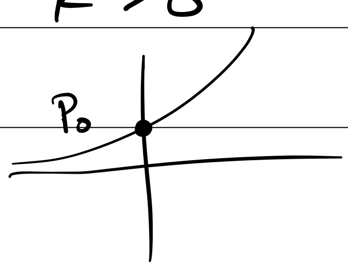
Growth rate: $k \approx 0.2299 \rightarrow \sim 23\%$

$$P(t) = P_0 e^{0.2299t}$$

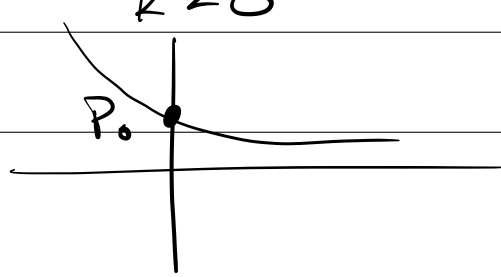
$$\Rightarrow 400 = P_0 e^{0.2299(3)}$$

$P_0 \approx 200.69 \rightarrow 201$ bacteria

$k > 0$



$k < 0$



Example: A thermometer is taken from an inside room to the outside, where the air temperature is 5° F. After 1 minute the thermometer reads 55° F, and after 5 minutes it reads 30° F. What is the initial temperature of the inside room?

$$\frac{dT}{dt} = k(T - T_m) \Rightarrow \frac{dT}{T - T_m} = k\,dt$$

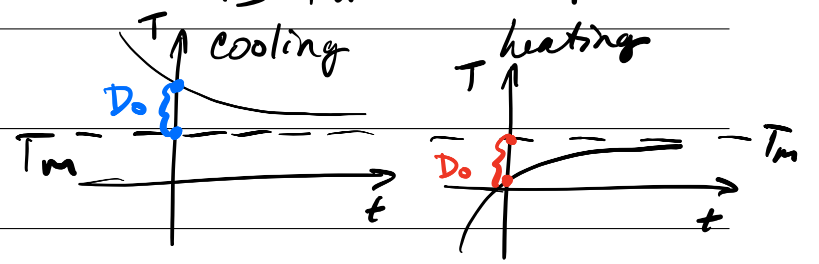
$$\ln|T - T_m| = kt + C \Rightarrow T - T_m = e^{kt+C}$$

$$T - T_m = D_0 e^{kt}$$

D_0 is initial temp difference

$$T(t) = T_m + D_0 e^{kt}$$

$k < 0$



$T_m = 5$ $\quad T(1) = 55$ $\quad T(5) = 30$

$$\hookrightarrow T(t) = 5 + D_0 e^{kt}$$

$$55 = 5 + D_0 e^{k} \qquad 50 = D_0 e^{k}$$

$$30 = 5 + D_0 e^{5k} \qquad 25 = D_0 e^{5k}$$

$$2 = e^{-4k} \Rightarrow k = -\frac{1}{4}\ln 2 \approx -0.1733$$

$$T(t) = 5 + D_0 e^{-0.1733t}$$

$$55 = 5 + D_0 e^{-0.1733} \Rightarrow D_0 = 50e^{0.1733}$$

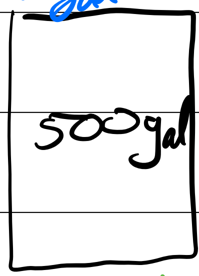
$$D_0 \approx 59.5 \qquad \text{Recall: } D_0 = T_0 - T_m$$

Since $T_m = 5$, initial thermometer temp is $59.5° + 5° = 64.5°$ F.

$A(0) = 0$

Example: A large tank is filled to capacity with 500 gallons of pure water. Brine containing 2 pounds of salt per gallon is pumped into the tank at a rate of 5 gal/min. The well-mixed solution is pumped out at the same rate. Find the number $A(t)$ of pounds of salt in the tank at time t.

$\frac{2 \text{ lb}}{\text{gal}} \cdot \frac{5 \text{ gal}}{\text{min}} = 10 \frac{\text{lb}}{\text{min}}$



$$\frac{dA}{dt} = \text{rate salt in} - \text{rate salt out}$$

$$\frac{dA}{dt} = 10 - \frac{1}{100} A$$

$\frac{A \text{ lb}}{\cancel{500}_{100} \text{ gal}} \cdot \frac{\cancel{5}^{1} \text{ gal}}{\text{min}}$

$$\frac{dA}{dt} + \frac{1}{100} A = 10$$

$\ldots$

$$A(t) = 1000 - 1000 e^{-\frac{1}{100} t}$$

$A(0)=0$, Tank: 500 gal   Rate in: 10 lb/gal

Example: Solve the previous problem under the assumption that the solution is pumped out a a faster rate of 10 gal/min. When is the tank empty?

Tank empty: 5 gal in − 10 gal out = losing 5 gal/min

empty in 100 min.

$\frac{2\text{ lb}}{\text{gal}} \cdot \frac{5\text{ gal}}{\text{min}} \to 10\text{ lb/gal}$

$$\frac{dA}{dt} = \text{rate salt in} - \text{rate salt out}$$

$$\frac{dA}{dt} = 10 - \frac{2A}{100-t}$$

$$\frac{A\text{ lb}}{(\cancel{500}^{\,100}-\cancel{5}^{\,1}t)\text{ gal}} \cdot \frac{\cancel{10}^{\,2}\text{ gal}}{\text{min}}$$

$$\frac{dA}{dt} + \frac{2}{100-t}A = 10$$

...

$$A(t) = 1000 - 10t - \frac{1}{10}(100-t)^2, \quad 0 \le t \le 100$$

Example: A 200-volt electromotive force is applied to an RC-series circuit in which the resistance is 1000 ohms and the capacitance is 5×10^{-6} farad. Find the charge $q(t)$ on the capacitor if $i(0) = 0.4$. Determine the charge and current at $t = 0.005$ s. Determine the charge as $t \to \infty$.

$$R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

$$C = 5 \times 10^{-6} = \frac{5}{10^6}$$

$$\frac{1}{C} = \frac{1}{5} \times 10^6 = \frac{10^6}{5}$$

$$1000q' + \frac{10^6}{5}q = 200$$

$$q' + 200q = \frac{1}{5}$$

$$\mu = e^{200t}$$

$$\frac{d}{dt}\left(e^{200t}q\right) = \frac{1}{5}e^{200t}$$

$$e^{200t}q = \frac{1}{1000}e^{200t} + C$$

$$q(t) = \frac{1}{1000} + Ce^{-200t}$$

As $t \to \infty$, $q \to \frac{1}{1000}$ coulomb

$$i = \frac{dq}{dt} = -200Ce^{-200t}$$

$$i(0) = 0.4 \Rightarrow 0.4 = -200C$$

$$C = -\frac{0.4}{200} = -\frac{4}{2000} = -\frac{1}{500}$$

$$C = -\frac{1}{500}$$

$$q(t) = \frac{1}{1000} - \frac{1}{500}e^{-200t}$$

$q(0.005)$, $i(0.005)$

coulombs amps