3.2 - Nonlinear Models

(used with large populations or when environmental factors inhibit growth)

Logistic growth: $\frac{dP}{dt} = P(a - bP)$

$= aP - bP^2$

Note: $-bP^2$ $(b>0)$ is called an inhibition or competition term.

Justification for the model:

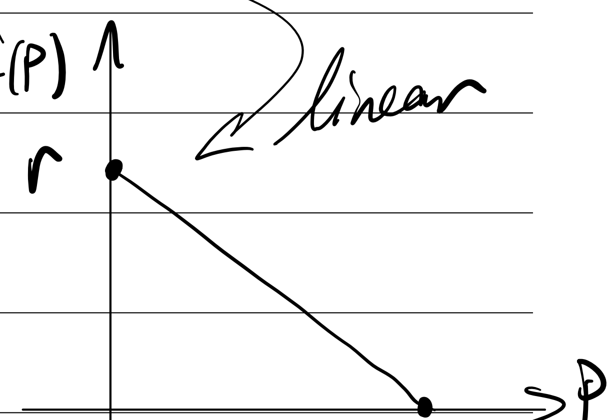
$\frac{dP/dt}{P} = k$ - Constant relative growth rate (3.1)

$\frac{dP/dt}{P} = f(P)$ - variable growth rate but still depends on the population

If K is the carrying capacity for the population (max pop)

$f(K) = 0$. If we let $f(0) = r$, then in the simplest case

linear



$f(P) = r - \frac{r}{K}P$

$$\frac{dP}{dt} = P\left(r - \frac{r}{K}P\right)$$

Relabeling gives $\frac{dP}{dt} = P(a - bP)$ $K = \frac{a}{b}$

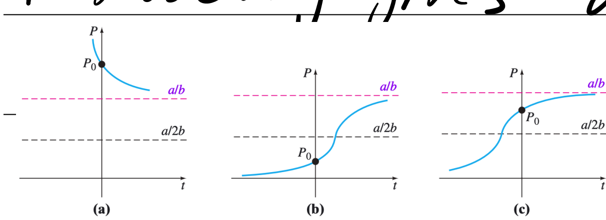


FIGURE 3.2.2 Logistic curves for different initial conditions

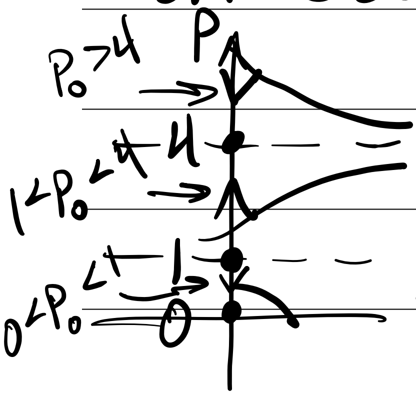
Modified logistic growth: $\dfrac{dP}{dt} = P(a - bP) \pm h$ restocking harvesting

Example: (a) If a constant number h of fish are harvested from a fishery per unit time, then a model for the population $P(t)$ of the fishery at time t is given by $\dfrac{dP}{dt} = P(a - bP) - h$, $P(0) = P_0$, where a, b, h, and P_0 are positive constants. Suppose $a = 5$, $b = 1$, and $h = 4$. Since the DE is autonomous, use the phase portrait concept of Section 2.1 to sketch representative solution curves corresponding to the cases $P_0 > 4$, $1 < P_0 < 4$, and $0 < P_0 < 1$. Determine the long-term behavior of the population in each case.

$$\frac{dP}{dt} = P(5-P) - 4 \Rightarrow \frac{dP}{dt} = -P^2 + 5P - 4$$

$$\frac{dP}{dt} = -(P^2 - 5P + 4) = -(P-4)(P-1)$$

critical numbers: $P = 4, 1$ (equilibrium solutions)



If $P_0 > 4$, P will decline to 4

If $1 < P_0 < 4$, P will increase to 4

If $0 < P_0 < 1$, the population will go extinct.

(b) Solve the IVP in part (a). Verify the results of your phase portrait in part (a) by using a graphing utility to plot the graph of $P(t)$ with an initial condition taken from each of the three intervals given.

$$\frac{dP}{dt} = -(P-4)(P-1) \Rightarrow -\frac{dP}{(P-4)(P-1)} = dt$$

PFD: $\ln\left|\dfrac{P-1}{P-4}\right| = 3t + C_1$

$$\frac{P-1}{P-4} = Ce^{3t} \qquad P(0) = P_0$$

$$\boxed{C = \frac{P_0 - 1}{P_0 - 4}}$$

$$P - 1 = (P-4)\,Ce^{3t}$$

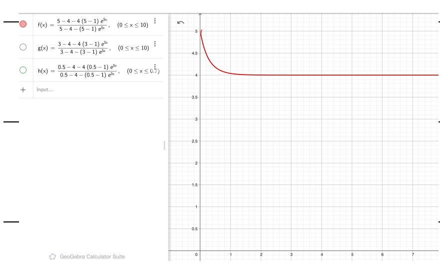
$$P - 1 = PCe^{3t} - 4Ce^{3t}$$

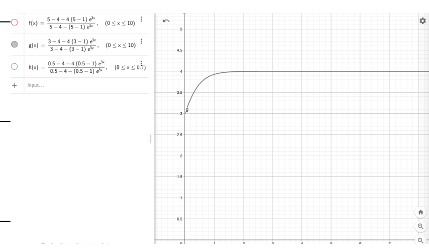
$$(1 - Ce^{3t})P = 1 - 4Ce^{3t}$$

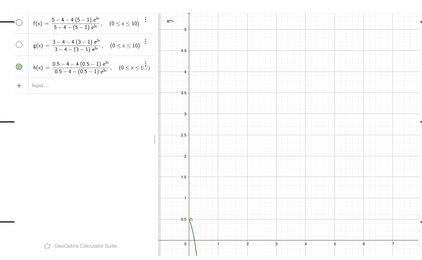
$$P = \frac{1 - 4\boxed{C}e^{3t}}{1 - \boxed{C}e^{3t}}$$

$$\vdots$$

$$P(t) = \frac{P_0 - 4 - 4(P_0 - 1)e^{3t}}{P_0 - 4 - (P_0 - 1)e^{3t}}$$







Extinct: $P = 0$ if $P_0 - 4 - 4(P_0 - 1)e^{3t} = 0$

we find $t = -\frac{1}{3}\ln\left[\frac{4(P_0 - 1)}{P_0 - 4}\right]$

(c) Use the information in parts (a) and (b) to determine whether the fishery population becomes extinct in finite time. If so, find that time.

Second-order chemical reaction: $\dfrac{dX}{dt} = k(\alpha - X)(\beta - X)$,
where $\alpha = \dfrac{a(M+N)}{M}$ and $\beta = \dfrac{b(M+N)}{N}$,
$a =$ # grams of chemical A,
$b =$ # grams of chemical B, and
$X(t)$ is grams of C, formed from M parts of A and N parts of B

Example: Two chemicals A and B are combined to form a chemical C. The rate, or velocity, of the reaction is proportional to the product of the instantaneous amounts of A and B not converted to chemical C. Initially, there are 80 grams of A and 50 grams of B, and for each gram of B, 2 grams of A is used. It is observed that 10 grams of C is formed in 10 minutes. How much is formed in 40 minutes? What is the limiting amount of C after a long time? How much of chemicals A and B remains after a long time?