

4.1 - Preliminary Theory-Linear Equations

Definition: The equations

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = 0 \quad (1) \quad \text{and}$$

$$a_n(x) \frac{d^n y}{dx^n} + a_{n-1}(x) \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x) \frac{dy}{dx} + a_0(x)y = g(x) \quad (2)$$

are n th-order, linear differential equations. (1) is **homogeneous**, and if $g(x) \neq 0$, then (2) is **nonhomogeneous**. *Abbr: hom, homog.*

Definition: The points $y(x_0) = y_0, y'(x_0) = y_1, \dots, y^{(n-1)}(x_0) = y_{n-1}$ are **initial conditions**. An **initial-value problem** is (1) or (2), together with initial conditions. In the event that we have information such as $y(a) = y_0, y(b) = y_1$, then we have a **boundary-value problem**.

Theorem: Existence of a Unique Solution

Let $a_n(x), a_{n-1}(x), \dots, a_1(x)$ and $g(x)$ be continuous on an interval I and let $a_n(x) \neq 0$ for every x in this interval. If $x = x_0$ is any point in this interval, then a solution $y(x)$ of the initial-value problem (1) exists on the interval and is unique.

These two important criteria will be assumed henceforth in the statement of theorems about linear DEs

A BVP can have zero, one, or infinitely many solutions.

Example: The given two-parameter family is a solution of the indicated differential equation on the interval $(-\infty, \infty)$. Determine whether a member of the family can be found that satisfies the boundary conditions.

$$y = c_1 x^2 + c_2 x^4 + 3; \quad x^2 y'' - 5x y' + 8y = 24$$

$$(a) \quad y(-1) = 0, \quad y(1) = 4$$

$$(b) \quad y(0) = 1, \quad y(1) = 2$$

$$(c) \quad y(0) = 3, \quad y(1) = 0$$

$$(d) \quad y(1) = 3, \quad y(2) = 15$$

$$a) y(-1) = 0 \Rightarrow 0 = c_1 + c_2 + 3 \Rightarrow c_1 + c_2 = -3 \quad \text{No.}$$

$$y(1) = 4 \Rightarrow 4 = c_1 + c_2 + 3 \Rightarrow c_1 + c_2 = 1$$

$$b) y(0) = 1 \Rightarrow 1 = 3 \quad \text{No}$$

$$c) y(0) = 3 \Rightarrow 3 = 3$$

$$y(1) = 0 \Rightarrow 0 = c_1 + c_2 + 3 \Rightarrow c_2 = -c_1 - 3$$

$$\text{Solution has form } y = c_1 x^2 + (-c_1 - 3)x^4 + 3$$

for $c_1 \in \mathbb{R}$. yes.

d) we'll find there's a unique solution.
(you do when you make time for it)

Definition: A set of functions $f_1(x), f_2(x), \dots, f_n(x)$ is said to be **linearly dependent** on an interval I if there exist constants $c_1, c_2, \dots, c_n$, not all zero, such that $c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x) = 0$ for every x in the interval. If the set of functions is not linearly dependent on the interval, it is said to be **linearly independent**.

Dep: $f_1(x) = 2x^2, f_2(x) = -4x^2$
 $c_1(2x^2) + c_2(-4x^2) = 0$
 Ind: e^{2x}, e^{3x}

Definition: Suppose each of the functions $f_1(x), f_2(x), \dots, f_n(x)$ possesses at least $n - 1$ derivatives. The determinant

$$W(f_1, f_2, \dots, f_n) = \begin{vmatrix} f_1 & f_2 & \dots & f_n \\ f_1' & f_2' & \dots & f_n' \\ \vdots & \vdots & \dots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \dots & f_n^{(n-1)} \end{vmatrix},$$

where the primes denote derivatives, is called the **Wronskian** of the functions

Theorem: Criterion for Linearly Independent Solutions

Let $y_1, y_2, \dots, y_n$ be n solutions of the homogeneous n th-order differential equation (1) on an interval I . Then the set of solutions is **linearly independent** on I if and only if $W(y_1, y_2, \dots, y_n) \neq 0$ for every x in the interval.

Example: Determine whether the given set of functions is linearly independent on the interval $(-\infty, \infty)$.

$$f_1(x) = \cos 2x, f_2(x) = 1, f_3(x) = \cos^2 x$$

2x2 determinant : $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

3x3 determinant : (expansion by minors)

$$\begin{vmatrix} +a & -b & +c \\ -d & +e & -f \\ +g & -h & +i \end{vmatrix} = +a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

$$W(f_1, f_2, f_3) = \begin{vmatrix} \cos 2x & 1 & \cos^2 x \\ -2\sin 2x & 0 & -2\cos x \sin x \\ -4\cos 2x & 0 & -2\cos 2x \end{vmatrix}$$

$$= -1 \begin{vmatrix} -2\sin 2x & -\sin 2x \\ -4\cos 2x & -2\cos 2x \end{vmatrix} = -(-4\sin 2x \cos 2x - 4\sin 2x \cos 2x) = 0$$

$$\Rightarrow \cos 2x = 2\cos^2 x - 1 \Rightarrow \cos 2x - 2\cos^2 x + 1 = 0$$

Theorem: Superposition Principle-Homogeneous Equations

Let $y_1, y_2, \dots, y_k$ be solutions of the homogeneous n th-order differential equation (1) on an interval I . Then the linear combination $c_1 y_1(x) + c_2 y_2(x) + \dots + c_k y_k(x)$, where the $c_i, i = 1, 2, \dots, k$ are arbitrary constants, is also a solution on the interval.

What about $y = e^{3x} + e^{-5x}$?

$$y' = 3e^{3x} - 5e^{-5x}, \quad y'' = 9e^{3x} + 25e^{-5x}$$

verify:

$$y = 1e^{3x} \Rightarrow y' = 1 \cdot 3e^{3x} \Rightarrow y'' = 1 \cdot 9e^{3x}$$

$$1 \cdot 9e^{3x} - 1 \cdot 2(3e^{3x}) - 1 \cdot 5e^{3x} = 1 \cdot 0 \quad \checkmark$$

What about $y = 1e^{3x}$?

$y_1 = c_1 e^{3x}$ is a solution $\rightarrow y_2 = c_2 e^{-5x}$

Ex: $y_1 = e^{3x}$ and $y_2 = e^{-5x}$ are solutions to $y'' + 2y' - 15y = 0$.

So $y = c_1 e^{3x} + c_2 e^{-5x}$ is also a solution for any $c_1, c_2 \in \mathbb{R}$.

Definition: Any set $y_1, y_2, \dots, y_n$ of n linearly independent solutions of the homogeneous n th-order differential equation (1) on an interval I is said to be a **fundamental set of solutions** on the interval.

Theorem: General Solution-Homogeneous Equations

Let $y_1, y_2, \dots, y_n$ be a fundamental set of solutions of the homogeneous linear n th-order differential equation (1) on an interval I . Then the **general solution** of the equation on the interval is $y = c_1 y_1(x) + c_2 y_2(x) + \dots + c_n y_n(x)$.

Example: Verify that the given functions form a fundamental set of solutions of the differential equation on the indicated interval. Form the general solution.

$$x^2 y'' + x y' + y = 0; \quad \cos(\ln x), \sin(\ln x), (0, \infty)$$

→ ① Plug in & show eqn is true

② show $W(f_1, f_2) \neq 0$

$$y = c_1 \cos(\ln x) + c_2 \sin(\ln x)$$

Theorem: Existence of a Fundamental Set

There exists a fundamental set of solutions for the homogenous linear n th-order differential equation (1) on an interval I .

Definition: Any function y_p , free of arbitrary parameters, that satisfies (2) is said to be a **particular solution** of the equation.

Theorem: General Solution–Nonhomogeneous Equations

Let y_p be any particular solution of the nonhomogeneous linear n th-order differential equation (2) on an interval I and let $y_1, y_2, \dots, y_n$ be a fundamental set of solutions of the associated homogeneous differential equation (1) on I . Then the **general solution** of the equation on the interval is

$y = c_1 y_1(x) + c_2 y_2(x) + \dots + c_n y_n(x) + y_p(x)$, where the $c_i, i = 1, 2, \dots, n$ are arbitrary constants.

Theorem: Superposition Principle–Nonhomogeneous Equations

Let $y_{p_1}, y_{p_2}, \dots, y_{p_k}$ be k particular solutions of the nonhomogeneous linear n th-order differential equation (2) on an interval I corresponding, in turn, to k distinct functions $g_1, g_2, \dots, g_k$. Then $y_p(x) = y_{p_1}(x) + y_{p_2}(x) + \dots + y_{p_k}(x)$ is a particular solution of

slap 'em together

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \dots + a_1(x)y' + a_0(x)y = g_1(x) + g_2(x) + \dots + g_k(x)$$

Example: (a) Verify that $y_{p1} = 3e^{2x}$ and $y_{p2} = x^2 + 3x$, respectively are particular solutions of

$$y'' - 6y' + 5y = -9e^{2x} \text{ and}$$

$$y'' - 6y' + 5y = 5x^2 + 3x - 16.$$

$$y_{p1}' = 6e^{2x}, \quad y_{p1}'' = 12e^{2x}$$

$$12e^{2x} - 36e^{2x} + 15e^{2x} = -9e^{2x} \quad \checkmark$$

$$y_{p2}' = 2x + 3, \quad y_{p2}'' = 2$$

$$2 - 12x - 18 + 5x^2 + 15x = 5x^2 + 3x - 16 \quad \checkmark$$

y_p

(b) Use part (a) to find the particular solution of

$$y'' - 6y' + 5y = 5x^2 + 3x - 16 - 9e^{2x}.$$

$$y_p = x^2 + 3x + 3e^{2x}$$

Solution to $y'' - 6y' + 5y = 0$ is

$$y_c = c_1 e^x + c_2 e^{5x}$$

complementary solution

General solution: $y = y_c + y_p$

$$y = c_1 e^x + c_2 e^{5x} + x^2 + 3x + 3e^{2x}$$