4.2 - Reduction of Order

Consider a 2nd-order DE

$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$ (1)

If y_1, y_2 form a fundamental set of solutions, then they are lin. indep.

So if $c_1 y_1 + c_2 y_2 = 0$ with c_1 and $c_2 \neq 0$, then we can write $y_2 = \boxed{-\frac{c_1}{c_2}} y_1$, where $-\frac{c_1}{c_2}$ is a function of x.

Then $y_2 = u(x) y_1$. unknown function of x

In standard form, (1) becomes

$$y'' + P(x) y' + Q(x) y = 0$$

If y_1 is a known solution, consider $y_2 = u y_1$

$y_2' = u' y_1 + u y_1'$, $y_2'' = u'' y_1 + 2u' y_1' + u y_1''$

$u'' y_1 + 2u' y_1' + u y_1'' + P(x) u' y_1 + P(x) u y_1' + Q(x) u y_1 = 0$

$u(y_1'' + P(x) y_1' + Q(x) y_1) = 0$ (y_1 is a solution)

Then $u'' y_1 + 2u_1' y_1' + P u' y_1 = 0$

$$u'' y_1 + (2y_1' + P y_1) u' = 0$$

Let $w = u'$ $\Rightarrow$ $w' = u''$

$w' y_1 + (2y_1' + P y_1) w = 0$ (linear)

$w' + \left(2 \frac{y_1'}{y_1} + P\right) w = 0$

$\mu = e^{\int \left(2 \frac{y_1'}{y_1} + P\right) dx} = e^{\ln y_1^2} e^{\int P dx}$

$= y_1^2 e^{\int P dx}$

$\frac{d}{dx}\left(y_1^2 e^{\int P dx} w \right) = 0$

$y_1^2 e^{\int P dx} w = c$ Let $c = 1$

$u' = w = \frac{1}{y_1^2} e^{-\int P dx}$

$u = \int \frac{e^{-\int P dx}}{y_1^2} dx$

The reduction of order formula is

$$y_2 = y_1 \int \frac{e^{-\int P(x) dx}}{y_1^2} dx$$

Example: The indicated function $y_1(x)$ is a solution of the given differential equation. Use reduction of order (the process or the formula) to find a second solution $y_2(x)$.

$y'' + 9y = 0; \quad y_1 = \sin 3x$

Process:

$y_2 = u(x)\,y_1$. Here, $y_2 = u\sin 3x$

$y_2' = u'\sin 3x + 3u\cos 3x$

$y_2'' = u''\sin 3x + 3u'\cos 3x + 3u'\cos 3x - 9u\sin 3x$

$y_2'' = u''\sin 3x + 6u'\cos 3x - 9u\sin 3x$

$u''\sin 3x + 6u'\cos 3x - \cancel{9u\sin 3x} + \cancel{9u\sin 3x} = 0$

$u''\sin 3x + 6u'\cos 3x = 0$

Let $w = u' \Rightarrow w' = u''$

$w'\sin 3x + 6w\cos 3x = 0$ (linear, separable)

$w' + 6\dfrac{\cos 3x}{\sin 3x}w = 0$

$\mu = e^{6\int \frac{\cos 3x}{\sin 3x}dx} = \sin^2 3x$

$\dfrac{d}{dx}(\sin^2 3x\; w) = 0$

$\sin^2 3x\; w = c$

$u' = w = c\csc^2 3x$

$u = c\int \csc^2 3x \Rightarrow u = -\frac{1}{3}c\cot 3x + C_1$

$y_2 = \cot 3x\; y_1$

$y_2 = u y_1 \Rightarrow y_2 = -\frac{1}{3}c\cot 3x\sin 3x = \cancel{-\frac{1}{3}c}\cos 3x$

$y = c_1\sin 3x + c_2\boxed{\cos 3x}$ $\qquad y_2 = \cos 3x$

Example: $6y'' + y' - y = 0$; $\quad y_1 = e^{x/3}$

Formula: ① Std form.

$y'' + \frac{1}{6}y' - \frac{1}{6}y = 0 \qquad P(x) = \frac{1}{6}$

$$y_2 = y_1 \int \frac{e^{-\int P\,dx}}{y_1^{2}}\,dx$$

$$= y_1 \int \frac{e^{-\int 1/6\,dx}}{e^{2x/3}}\,dx$$

$$= y_1 \int \frac{e^{-x/6}}{e^{2x/3}}\,dx = y_1 \int e^{-5x/6}\,dx$$

$$= y_1 \left(-\frac{6}{5} e^{-5x/6} + C \right)$$

we don't need these

$$y_2 = e^{x/3} e^{-5x/6} = e^{-x/2}$$

Example: $x^2y'' - 3xy' + 5y = 0; \quad y_1 = x^2\cos(\ln x)$

Example: The indicated function$y_1(x)$ is a solution of the associated homogeneous equation. Use the method of reduction of order [the process] to find a second solution $y_2(x)$ of the homogeneous equation and a particular solution $y_p(x)$ of the given nonhomogeneous equation.

$y'' - 4y' + 3y = x; \quad y_1 = e^x$

Starts with $y_2 = u\, y_1$

But we find $y_2 + y_p = u\, y_1$